Midterm 2 review
Math 265
Fall 2007

§13.3. Arc Length and Curvature.

Assume that the curve C is described by the vector-valued function $\mathbf{r}(r) = \langle f(t), g(t), h(t)\rangle$, and that C is traversed exactly once as t ranges from a to b. If $\mathbf{r}(t)$ is piecewise smooth, then the *length of* C is

$$L = \int_a^b |\mathbf{r}'(t)|\, dt = \int_a^b \sqrt{[f(t)]^2 + [g(t)]^2 + [h(t)]^2}\, dt.$$

The *arc length function* for $\mathbf{r}(t)$ is

$$s(t) = \int_a^t |\mathbf{r}'(u)|\, du = \int_a^t \sqrt{[f(u)]^2 + [g(u)]^2 + [h(u)]^2}\, du.$$

The Fundamental Theorem of Calculus implies that $s(t)$ is differentiable and $s'(t) = |\mathbf{r}'(t)|$.

The curve C is *parametrized with respect to arc length* if $|\mathbf{r}'(t)| = 1$.

To find a parametrization of C with respect to arc length, take the following steps.

1. Find the arc length function $s(t)$.
2. Set $s = s(t)$ and solve for t in terms of s: so $s = s(t)$.
3. Set $\mathbf{r}_1(s) = \mathbf{r}(t(s))$. This is the parametrization by arc length.

The *curvature* of C is $\kappa = |d\mathbf{T}/ds|$ where $\mathbf{T}$ is the unit tangent vector

$$\mathbf{T}(t) = \frac{1}{|\mathbf{r}'(t)|}\mathbf{r}'(t).$$

In practice, we compute κ with the following formulas:

$$\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

The *unit normal vector* for $\mathbf{r}(t)$ is

$$\mathbf{N}(t) = \frac{1}{|\mathbf{T}'(t)|}\mathbf{T}'(t).$$

The *binormal vector* for $\mathbf{r}(t)$ is

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t).$$

These vectors satisfy the following properties:

$$|\mathbf{T}(t)| = 1 \qquad |\mathbf{N}(t)| = 1 \qquad |\mathbf{B}(t)| = 1$$
$$\mathbf{T}(t)\cdot\mathbf{N}(t) = 0 \qquad \mathbf{T}(t)\cdot\mathbf{B}(t) = 0 \qquad \mathbf{N}(t)\cdot\mathbf{B}(t) = 0$$

The *normal plane* for the curve at the point $P(f(t_0), g(t_0), h(t_0))$ is the plane passing through P with normal vector $\mathbf{T}(t)$.

The *osculating plane* for the curve at P is is the plane passing through P with normal vector $\mathbf{B}(t)$.

§13.4. Motion in Space: Velocity and Acceleration.

The motion of a particle in space is described by $\mathbf{r}(r) = \langle f(t), g(t), h(t)\rangle$. The velocity, speed and acceleration of the particle are, respectively,

$$\begin{aligned}
&\mathbf{v}(t) = \mathbf{r}'(t)\\
&v(t) = |\mathbf{v}(t)| = |r'(t)| = s'(t)\\
&\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)
\end{aligned}$$

Acceleration can be written in terms of the unit normal vector and the binormal vector:

$$\mathbf{a}(t) = v'(t)\mathbf{T}(t) + [v(t)]^2\kappa(t)\mathbf{N}(t).$$

The *tangential component of* $\mathbf{a}(t)$ is

$$a_T = v'(t) = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|}$$

and the *normal component of* $\mathbf{a}(t)$ is

$$a_N = [v(t)]^2\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$$

and we have $\mathbf{a}(t) = a_T\mathbf{T}(t) + a_N\mathbf{N}(t)$.

§14.1. Functions of Several Variables.

A *function of two variables* is a rule of assignment f where
Input: an ordered pair (x, y) of real numbers, and
Output: a real number $f(x, y)$.

The *domain* of f is the set of all allowable inputs. The *range* of f is the set of all outputs. The *graph* of f is the set of all points (x, y, z) in 3-space such that $z = f(x, y)$. The *level curves* of f are the curves $f(x, y) = k$ where k is a constant in the range of f. A *contour map* of f is a graph of several level curves.

A *function of three variables* is a rule of assignment f where
Input: an ordered triple (x, y, z) of real numbers, and
Output: a real number $f(x, y, z)$.

The *domain* of f is the set of all allowable inputs. The *range* of f is the set of all outputs. The *level surfaces* of f are the surfaces $f(x, y, z) = k$ where k is a constant in the range of f.

§14.2. Limits and Continuity.

Let f be a function of two variables x and y, and let L be a real number. The *limit* of $f(x, y)$ as (x, y) approaches (a, b) is L if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x, y) - L| < \epsilon$ whenever (x, y) is in the domain of f and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$. In other words, we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

when $f(x)$ gets arbitrarily close to L as (x, y) gets arbitrarily close to (x, y) from any direction.

$\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist under the following conditions: There exist two paths to (a, b) such that $f(x, y) \to L_1$ along the first path and $f(x, y) \to L_2$ along the second path where $L_1 \neq L_2$.

The function f is *continuous* at (a, b) if $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$, that is, when the following conditions are satisfied:

1. $f(a, b)$ exists;
2. $\lim_{(x,y)\to(a,b)} f(x, y)$ exists; and
3. $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

Every polynomial in two variables is continuous at every point (a, b). Every rational function in two variables is continuous at every point in its domain, that is, at every point where the denominator is nonzero. If $g(x, y)$ is continuous at the point (a, b) and $h(t)$ is continuous at the point $g(a, b)$, then the composition $f(x, y) = h(g(x, y))$ is continuous at the point (a, b).

§14.3. Partial Derivatives.

Let f be a function of two variables x and y. The *partial derivative of f with respect to x* at the point (a, b) is

$$f_x(a, b) = \frac{\partial f}{\partial x} = \frac{d}{dx}[f(x, b)]\bigg|_{x=a}$$

and the *partial derivative of f with respect to y* at the point (a, b) is

$$f_y(a, b) = \frac{\partial f}{\partial y} = \frac{d}{dy}[f(a, y)]\bigg|_{y=b}.$$

To compute $f_x(x, y)$, treat y as a constant and take the derivative with respect to x. To compute $f_y(x, y)$, treat x as a constant and take the derivative with respect to y.

The *second order partial derivatives of f* are

$$f_{xx}(x,y) = (f_x)_x(x,y) \qquad f_{yx}(x,y) = (f_y)_x(x,y)$$
$$f_{xy}(x,y) = (f_x)_y(x,y) \qquad f_{yy}(x,y) = (f_y)_y(x,y).$$

Higher order partial derivatives like $f_{xyyx}(x,y)$ are defined similarly.

Clairaut's Theorem: If f is defined on a disk D containing the point (a,b) and the functions f_{xy} and f_{yx} are continuous on D, then $f_{xy}(a,b) = f_{yx}(a,b)$.

Partial derivatives for functions of three variables are defined similarly:

$$f_x(a,b,c) = \frac{\partial f}{\partial x} = \frac{d}{dx}[f(x,b,c)]\Big|_{x=a}$$
$$f_y(a,b,c) = \frac{\partial f}{\partial y} = \frac{d}{dy}[f(a,y,c)]\Big|_{y=b}$$
$$f_z(a,b,c) = \frac{\partial f}{\partial z} = \frac{d}{dz}[f(a,b,z)]\Big|_{z=c}$$

and so on

§14.4. Tangent Planes and Linear Approximations.

Let S be the surface $z = f(x,y)$ and assume that f_x and f_y are continuous at (x_0,y_0). Consider the point $P(x_0,y_0,z_0)$ where $z_0 = f(x_0,y_0)$. The *tangent plane* to S at the point P is described by the equation

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

The *linear approximation* of f at (x_0,y_0) is

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0).$$

The function f is *differentiable* at (x_0,y_0) if the linear approximation of f at (x_0,y_0) is a "good" approximation of f for (x,y) near to (x_0,y_0). See p. 926 of the text for a technical definition.

Criterion for differentiability: If the partial derivatives f_x and f_y are continuous at (x_0,y_0), then f is differentiable at (x_0,y_0).

The *differential* of f is

$$df = f_x(x,y)\,dx + f_y(x,y)\,dy.$$

§14.5. The Chain Rule.

Case 1. $z = f(x,y)$ and $x = g(t)$ and $y = h(t)$. Then $z = f(g(t), h(t))$ and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Case 2. $z = f(x,y)$ and $x = g(s,t)$ and $y = h(s,t)$. Then $z = f(g(s,t), h(s,t))$ and

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Case n. Assume $u = f(x_1, x_2, \ldots, x_n)$ and $x_1 = g_1(t_1, \ldots, t_m)$, $x_2 = g_2(t_1, \ldots, t_m)$, …, $x_n = g_n(t_1, \ldots, t_m)$. Then

$$u = f(g_1(t_1, \ldots, t_m), g_2(t_1, \ldots, t_m), \ldots, g_n(t_1, \ldots, t_m))$$

and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for $i = 1, \ldots, m$.

Implicit differentiation.

Case 1. x and y satisfy $F(x,y) = 0$. If F is differentiable and $F_y(x,y) \neq 0$, then

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)}$$

Case 2. x, y and z satisfy $F(x,y,z) = 0$. If F is differentiable and $F_z(x,y,z) \neq 0$, then

$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} \qquad \frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)}$$

§14.6. Directional Derivatives and the Gradient Vector.

Let f be a function of x and y. The *gradient* of f is

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle .$$

The *directional derivative* of f in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ at the point (x_0, y_0) is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

when the limit exists. If f is differentiable, then

$$D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x,y) \cdot \langle a, b \rangle .$$

Let f be a function of x, y and z. The *gradient* of f is

$$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle .$$

The *directional derivative* of f in the direction of a unit vector $\mathbf{u} = \langle a,b,c \rangle$ at the point (x_0,y_0,z_0) is

$$D_{\mathbf{u}}f(x_0,y_0,z_0) = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb, z_0+hc) - f(x_0,y_0,z_0)}{h}$$

when the limit exists. If f is differentiable, then

$$D_{\mathbf{u}}f(x_0,y_0,z_0) = \nabla f(x,y,z_0) \cdot \langle a,b,c \rangle .$$

Let f be a differentiable function of 2 or 3 variables. The maximal value of $D_{\mathbf{u}}f$ at a point is $|\nabla f|$, and it occurs in the direction of ∇f. The minimal value of $D_{\mathbf{u}}f$ at a point is $-|\nabla f|$, and it occurs in the direction of $-\nabla f$.

Let f be a differentiable function of x, y and z. The *tangent plane* to the level surface $f(x,y,z) = k$ at the point $P(x_0,y_0,z_0)$ has equation

$$\nabla f(x_0,y_0,z_0) \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0.$$

The *normal line* to the level surface at P is the line passing through the point P with direction vector $\nabla f(x_0,y_0,z_0)$.

§14.7. Maximum and Minimum Values.

Let f be a function of x and y.

f has a *local maximum* at (a,b) if $f(x,y) \leq f(a,b)$ for all points (x,y) in some disk centered at (a,b).

f has an *absolute maximum* at (a,b) if $f(x,y) \leq f(a,b)$ for all points (x,y) in the domain of f.

f has a *local minimum* at (a,b) if $f(x,y) \geq f(a,b)$ for all points (x,y) in some disk centered at (a,b).

f has an *absolute minimum* at (a,b) if $f(x,y) \geq f(a,b)$ for all points (x,y) in the domain of f.

First Derivative Test. If f has a local maximum or a local minimum at (a,b) and the partial derivatives $f_x(a,b)$ and $f_y(a,b)$ exist, then $f_x(a,b) = 0 = f_y(a,b)$.

The point (a,b) is a *critical point* for f if $f_x(a,b) = 0 = f_y(a,b)$.

Second Derivative Test. Assume that the second order partial derivatives of f are continuous in a disk with center (a,b). Assume that (a,b) is a critical point for f and set

$$D = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2 = \begin{vmatrix} f_{xx}(a,b) & f_{yx}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{vmatrix}$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum at (a, b).

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum at (a, b).

(c) If $D < 0$, then f has a saddle point at (a, b).

(d) If $D = 0$, then no conclusion may be drawn from the test.

A subset D of the plane is *closed* if it contains all its boundary points. The set D is *bounded* if it is contained in some disk.

If f is continuous on a closed, bounded regions D, then f has an absolute maximum and an absolute minimum in D. To find the absolute maximum and an absolute minimum, take the following steps.

1. Find the values of f at the critical points of f in D.

2. Find the extreme values of f on the boundary of D.

3. The largest value from steps 1 and 2 is the absolute maximum of f in D. The smallest value from steps 1 and 2 is the absolute minimum of f in D.

Be sure to review the sections of the text (especially the examples), your notes, your homework, and your quizzes.

Practice Exercises:
pp. 882–883: 6(c), 8, 10–13, 15, 17–20
pp. 975–977: 1–10, 13–17, 19–23, 25-29, 31–40, 42–48, 51–56, 63