Final exam review
Math 265
Fall 2007

This exam will be cumulative. Consult the review sheets for the midterms for reviews of Chapters 12–15.

§16.1. Vector Fields.

A vector field on $\mathbb{R}^2$ is a function $\mathbf{F}$ from $\mathbb{R}^2$ to V_2.
Input: a point (x, y)
Output: a vector $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$.

A graph of $\mathbf{F}$ can be obtained by plotting several vectors $\mathbf{F}(x, y)$ with initial point (x, y).

A vector field on $\mathbb{R}^3$ is a function $\mathbf{F}$ from $\mathbb{R}^3$ to V_3.
Input: a point (x, y, z)
Output: a vector $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$.

§16.2. Line Integrals.

Let C be a smooth curve in $\mathbb{R}^2$, parametrized by the vector-valued function $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$. Let f be a continuous real-valued function of two variables.

Line integral of f with respect to arc length:

$$\int_C f(x, y)\, ds = \int_{t=a}^{b} f(x(t), y(t))\sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

Line integral of f with respect to x:

$$\int_C f(x, y)\, dx = \int_{t=a}^{b} f(x(t), y(t))x'(t)\, dt$$

Line integral of f with respect to y:

$$\int_C f(x, y)\, dy = \int_{t=a}^{b} f(x(t), y(t))y'(t)\, dt$$

If $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ is a continuous vector field on $\mathbb{R}^2$, then

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C [P(x, y)\, dx + Q(x, y)\, dy] \\ &= \int_{t=a}^{b} [P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)]\, dt.\end{aligned}$$

Let C be a smooth curve in $\mathbb{R}^3$, parametrized by the vector-valued function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$. Let f be a continuous real-valued function of three variables.

Line integral of f with respect to arc length:

$$\int_C f(x,y,z)\,ds = \int_{t=a}^{b} f(x(t),y(t),z(t))\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\,dt$$

Line integral of f with respect to x:

$$\int_C f(x,y,z)\,dx = \int_{t=a}^{b} f(x(t),y(t),z(t))x'(t)\,dt$$

Line integral of f with respect to y:

$$\int_C f(x,y,z)\,dy = \int_{t=a}^{b} f(x(t),y(t),z(t))y'(t)\,dt$$

Line integral of f with respect to z:

$$\int_C f(x,y,z)\,dy = \int_{t=a}^{b} f(x(t),y(t),z(t))z'(t)\,dt$$

If $\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ is a continuous vector field on $\mathbb{R}^3$, then

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C [P(x,y,z)\,dx + Q(x,y,z) + R(x,y,z)\,dz] \\ &= \int_{t=a}^{b} [P(x(t),y(t),z(t))x'(t) + Q(x(t),y(t),z(t))y'(t) \\ &\qquad + R(x(t),y(t),z(t))z'(t)]\,dt.\end{aligned}$$

§16.3. The Fundamental Theorem for Line Integrals.

This is a version of the Fundamental Theorem of Calculus for line integrals: Let C be a smooth curve (in $\mathbb{R}^2$ or $\mathbb{R}^3$) parametrized by the vector-valued function $\mathbf{r}(t)$, $a \le t \le b$. Let f be a differentiable function (in two or three variables) such that ∇f is continuous. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

In particular, if C_1 is another smooth curve with the same initial and terminal points as C, then

$$\int_C \nabla f \cdot d\mathbf{r} = \int_{C_1} \nabla f \cdot d\mathbf{r}$$

A vector field $\mathbf{F}$ is conservative if there is a differentiable function f such that $\mathbf{F} = \nabla f$. We say that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if, for each pair of piecewise smooth curves C_1 and C_2 with the same initial and terminal points, we have $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

First test to see whether $\mathbf{F}$ is conservative: $\mathbf{F}$ is conservative if and only if $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path.

Second test to see whether $\mathbf{F}$ is conservative: Let $\mathbf{F}$ be a vector field on $\mathbb{R}^2$, say $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ where P and Q have continuous first order partial derivatives. Then $\mathbf{F}$ is conservative if and only if $P_y = Q_x$.

§16.4. Green's Theorem.

Let C be a simple closed curve in $\mathbb{R}^2$ with positive orientation. Let D be the planar region bounded by C. Let $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ be a vector field such that P and Q have continuous first order partial derivatives. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C [P(x, y)\, dx + Q(x, y)\, dy] = \iint_D (Q_x - P_y)\, dA.$$

This formula is useful, for instance, when C has several pieces and $Q_x - P_y$ is particularly easy to integrate.

§16.5. Curl and Divergence.

Write $\nabla = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z\rangle$.

Let $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$.

The curl of $\mathbf{F}$ is

$$\begin{aligned} \operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} \\ &= (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k} \end{aligned}$$

Third test to see whether $\mathbf{F}$ is conservative: If P, Q, and R have continuous first order partial derivatives, then $\mathbf{F}$ is conservative if and only if $\operatorname{curl} \mathbf{F} = \mathbf{0}$.

The divergence of $\mathbf{F}$ is

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = P_x + Q_y + R_z$$

If P, Q, and R have continuous second order partial derivatives, then $\operatorname{div} \operatorname{curl} \mathbf{F} = \mathbf{0}$.

Let C be a simple closed curve in $\mathbb{R}^2$ with positive orientation. Let D be the planar region bounded by C. Let $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ be a vector field such that P and Q have continuous first order partial derivatives.

Second version of Green's Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}\, dA.$$

Assume that C is parametrized by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ and set

$$\mathbf{n}(t) = \left\langle \frac{y'(t)}{|\mathbf{r}'(t)|}, -\frac{x'(t)}{|\mathbf{r}'(t)|} \right\rangle$$

Third version of Green's Theorem:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \operatorname{div} \mathbf{F}\, dA$$

§16.6. Parametric Surfaces and Their Areas.

Parametric equations:

$$x = f(u,v) \qquad y = g(u,v) \qquad z = h(u,v) \qquad (u,v) \text{ in } D$$

Vector equation:

$$\mathbf{r}(u,v) = \langle f(u,v), g(u,v), h(u,v) \rangle \qquad (u,v) \text{ in } D$$

Be able to identify a surface described parametrically. Also, given a cartesian equation for a surface, be able to describe it parametrically.

The tangent plane to a the parametric surface at the point $(x_0, y_0, z_0) = (f(u_0, v_0), g(u_0, v_0), h(u_0, v_0))$ is the plane passing through the point (x_0, y_0, z_0) with normal vector

$$\mathbf{n} = \mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0).$$

The surface area of the surface is

$$A = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA.$$

§16.7. Surface Integrals.

Let S be the surface describe parametrically by the vector-function $\mathbf{r}(u,v)$ for (u,v) in D. Assume that $\mathbf{r}$ has continuous first order partial derivatives. If f is a continuous real-falued function, then

$$\iint_S f(x,y,z)\, dS = \iint_D f(\mathbf{r}(u,v)) |\mathbf{r}_u \times \mathbf{r}_v|\, dA.$$

The unit normal vector to S is

$$\mathbf{n} = \frac{1}{|\mathbf{r}_u \times \mathbf{r}_v|} \mathbf{r}_u \times \mathbf{r}_v.$$

If is a continuous vector field on $\mathbb{R}^3$, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

§16.8. Stoke's Theorem.

Let S be a piecewise smooth parametrized surface such that ∂S is simple and closed. Let $F = \langle P, Q, R \rangle$ be a vector field such that P, Q and R have continuous first order partial derivatives. Then

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}.$$

This formula is useful, for instance, when ∂S has several pieces and the integral $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ is particularly easy to evaluate.

§16.9. The Divergence Theorem.

Let E be a simple solid region, and assume that the boundary surface ∂E is parametrized so that the unit normal $\mathbf{n}$ points away from E. Let $F = \langle P, Q, R \rangle$ be a vector field such that P, Q and R have continuous first order partial derivatives. Then

$$\iiint_E \operatorname{div} \mathbf{F} \, dV = \iint_{\partial E} \mathbf{F} \cdot d\mathbf{S}.$$

This formula is useful, for instance, when ∂E has several pieces and the integral $\iiint_E \operatorname{div} \mathbf{F} \, dV$ is particularly easy to evaluate.

Be sure to review the sections of the text (especially the examples), your notes, your homework, and your quizzes.

Practice Exercises:
pp. 1136–1137: 1–19, 21, 24, 25, 26(a,c), 27–40