MONOMIAL IDEALS: HOMEWORK 1

Exercise 1. (Fact I.3.7) Let R be a commutative ring with identity. Prove the following.

(a) Let $I = (f_1, \dots, f_n)R$ and let $J = (g_1, \dots, g_m)R$. Then $I + J$ is generated by the set $\{f_1, \dots, f_n, g_1, \dots, g_m\}$.

(b) Let n be a positive integer and let $S_1, S_2, \dots, S_n$ be subsets of R. Then $\sum_{j=1}^{n}(S_j)R = (\cup_{j=1}^{n} S_j)R$.

(c) If $\{S_\lambda\}_{\lambda\in\Lambda}$ is a collection of subsets of R, then $\sum_{\lambda\in\Lambda}(S_\lambda)R = (\cup_{\lambda\in\Lambda} S_\lambda)R$.

Exercise 2. (Fact I.3.10) Let R be a commutative ring with identity. Prove the following.

(a) (0 and 1) If I is an ideal of R, then $RI = I$ and $0I = 0$.

(b) (commutative law) If I and J are ideals of R, then $IJ = JI$. Moreover, if $I_1, \dots, I_n$ are ideals of R and $i_1, \dots, i_n$ is a permutation of the numbers $1, \dots, n$, then $I_1 \cdots I_n = I_{i_1} \cdots I_{i_n}$.

(c) (associative law) If I, J and K are ideals of R, then $(IJ)K = IJK = I(JK)$. (More general associative laws hold by induction on the number of ideals involved.)

(d) (distributive law) If I, J and K are ideals of R, then $(I + J)K = IK + JK = K(I + J)$. (More general distributive laws hold by induction on the number of ideals involved.)

Exercise 3. (Fact I.3.11) Let R be a commutative ring with identity. Prove the following.

(a) If $I = (f_1, \dots, f_n)R$ and $J = (g_1, \dots, g_m)R$, then

$$IJ = (\{f_i g_j \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\})R.$$

(b) If $I_j = (S_j)R$ for $j = 1, \dots, n$ then

$$I_1 \cdots I_n = (\{s_1 \cdots s_n \mid s_i \in S_i \text{ for } i = 1, \dots, n\})R.$$

(c) If n is a positive integer and $I = (f_1, \dots, f_m)R$, then

$$I^n = (\{f_{i_1} \cdots f_{i_n} \mid 1 \leqslant i_j \leqslant m \text{ for } j = 1, \dots, n\})R.$$

Exercise 4. (Proposition I.3.13) Let R be a commutative ring with identity. Let I, J, and K be ideals of R, and let $\{I_\lambda\}_{\lambda\in\Lambda}$ be a collection of ideals of R. Prove the following:

(a) There are containments $(I :_R J)J \subseteq I \subseteq (I :_R J)$.

(b) There is an equality $(\cap_{\lambda\in\Lambda} I_\lambda :_R J) = \cap_{\lambda\in\Lambda}(I_\lambda :_R J)$.

(c) There is an equality $(J :_R \sum_{\lambda\in\Lambda} I_\lambda) = \cap_{\lambda\in\Lambda}(J :_R I_\lambda)$.

(d) Given a subset $S \subseteq R$, an element $r \in R$ is in $(I :_R (S)R)$ if and only if $rs \in I$ for each $s \in S$.