MONOMIAL IDEALS: HOMEWORK 2

Exercise 1. (Fact I.3.6) Let R be a commutative ring with identity. Prove the following.

(a) (0 and 1) If I is an ideal in R, then $0 + I = I$ and $R + I = R$.

(b) (commutative law) If I and J are ideals of R, then $I + J = J + I$. More generally, if $\{I_\lambda\}_{\lambda\in\Lambda}$ is a collection of ideals of R and $f\colon \Lambda \to \Lambda$ is a bijection, then $\sum_{\lambda\in\Lambda} I_\lambda = \sum_{\lambda\in\Lambda} I_{f(\lambda)}$.

(c) (associative law) If I, J and K are ideals of R, then $(I+J)+K = I+J+K = I+(J+K)$. (More general associative laws, using more than three ideals, hold by induction on the number of ideals.)

Exercise 2. (Proposition I.3.18) Let R be a commutative ring with identity. Let n be a positive integer, and let $I, J, I_1, I_2, \ldots, I_n$ be ideals of R. statements:

(a) If $I \subseteq J$, then $\operatorname{rad}(I) \subseteq \operatorname{rad}(J)$.

(b) There are equalities $\operatorname{rad}(IJ) = \operatorname{rad}(I \cap J) = \operatorname{rad}(I) \cap \operatorname{rad}(J)$.

(c) There are equalities

$$\operatorname{rad}(I_1 I_2 \cdots I_n) = \operatorname{rad}(I_1 \cap I_2 \cap \cdots \cap I_n) = \operatorname{rad}(I_1) \cap \operatorname{rad}(I_2) \cap \cdots \cap \operatorname{rad}(I_n).$$

(d) $\operatorname{rad}(I + J) = \operatorname{rad}(\operatorname{rad}(I) + \operatorname{rad}(J))$.

(e) $\operatorname{rad}(I_1 + I_2 + \cdots + I_n) = \operatorname{rad}(\operatorname{rad}(I_1) + \operatorname{rad}(I_2) + \cdots + \operatorname{rad}(I_n))$.

Exercise 3. Let A be a commutative ring with identity and let $I \subseteq A$ be an ideal.

(a) Assume that I has the following property: there exists an element $f \in R$ such that f is not in I, but $f \in J$ for every ideal J of R that properly contains I. Prove that I is irreducible.

(b) Does the converse of part (a) hold? That is, if I is irreducible, must there exist an element $f \in R$ such that f is not in I, but $f \in J$ for every ideal J of R that properly contains I?

Exercise 4. Let A be a commutative ring with identity and let $I, J \subseteq A$ be ideals. Let P be a prime ideal of A and prove the following statements:

(a) If $IJ \subseteq P$ then either $I \subseteq P$ or $J \subseteq P$.

(b) If $I \cap J \subseteq P$ then either $I \subseteq P$ or $J \subseteq P$.

(c) P is an irreducible ideal.