MONOMIAL IDEALS: HOMEWORK 4

Exercise 1. Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Fix a monomial $z \in [R]$ and set $(\mathbf{X}) = (X_1, \ldots, X_d)R$.
(a) Prove that $\operatorname{rad}(\mathrm{P}_R(z)) = \operatorname{rad}((\mathbf{X}))$.
(b) Prove that, if A is a field, then $\operatorname{rad}(\mathrm{P}_R(z)) = \operatorname{rad}((\mathbf{X})) = (\mathbf{X})$.

Exercise 2. Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Let $z_1, \ldots, z_n$ be monomials in $[R]$ and $(\mathbf{X}) = (X_1, \ldots, X_d)R$. Prove that, if $I = \cap_{i=1}^{n} \mathrm{P}_R(z_i)$, then $\operatorname{rad}(I) = \operatorname{rad}((\mathbf{X}))$.

Exercise 3. Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Let f and g be monomials in R.
(a) Prove that if $f \in (g)R$, then $\deg(f) \geqslant \deg(g)$.
(b) Prove or disprove: If $\deg(f) \geqslant \deg(g)$, then $f \in (g)R$.
(c) Prove that if $\deg(f) = \deg(g)$ and $g \in (f)R$, then $g = f$.
(d) Prove that if $\deg(f) = \deg(g)$ and $f \neq g$, then $f \in \mathrm{P}_R(g)$.

Exercise 4. Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Let f be a monomial in R and let n be an integer such that $n > 1$. Prove that $\deg(f) < n$ if and only if there exists a monomial g of degree $n-1$ such that $g \in (f)R$.