MONOMIAL IDEALS: HOMEWORK 5

Exercise 1. Prove Theorem II.5.13(a): Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Set $(\mathbf{X}) = (X_1, \ldots, X_d)R$. If J is a monomial ideal in R, then $(J :_R (\mathbf{X})) = J + (\mathrm{C}_R(J))R$.

Exercise 2. Prove the following facts for use in Corollary II.5.3: Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Let J be a monomial ideal in R, and fix a monomial $w \in [R]$.

(a) If $w \notin J$, then $J \subseteq \mathrm{P}_R(w)$.
(b) If w is an J-corner element, then $J \subseteq \mathrm{P}_R(w)$.
(c) In the notation of Corollary II.5.3, we have $J \subseteq J'$.

Exercise 3. Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Set $(\mathbf{X}) = (X_1, \ldots, X_d)R$ and let J be a monomial ideal in R such that $\mathrm{rad}\,(J) = \mathrm{rad}\,((\mathbf{X}))$.

(a) Prove that, for each $n \geqslant 0$, the set of monomials $[R] \smallsetminus [J^n]$ is finite.
(b) Define the function $F \colon \mathbb{N} \to \mathbb{N}$ by the formula

$$F(n) = [\text{the number of monomials in } R - (\mathbf{X})^n] = |[R] \smallsetminus [(\mathbf{X})^n]|.$$

Prove that there is a polynomial $f \in \mathbb{Q}[x]$ of degree d such that $F(n) = f(n)$ for all $n \in \mathbb{N}$.

The function F is the *Hilbert function* of R associated to the ideal $(\mathbf{X})$, and f is the associated *Hilbert polynomial*.

A deeper result (one that we do not expect you to prove) says the following. Let $F_J \colon \mathbb{N} \to \mathbb{N}$ be the function

$$F(n) = [\text{the number of monomials in } R - J^n] = |[R] \smallsetminus [J^n]|.$$

There exists $n_0 \in \mathbb{N}$ and a polynomial $f_J \in \mathbb{Q}[x]$ of degree d such that $F_J(n) = f_J(n)$ for all $n \geqslant n_0$.

Exercise 4. Prove Corollary II.5.11: Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Let J be a monomial ideal in R and let $z \in \mathrm{C}_R(J)$. If $d \geqslant 2$, then the monomials $X_1 z, \ldots, X_d z$ are members of distinct principal ideals generated by monomials in J.