MONOMIAL IDEALS: HOMEWORK 8

Exercise 1. Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X, Y, Z]$ in three variables. Show that

$$(X^3, Y^2, XZ, Z^3)R = (X^3, Y^2, Z)R \cap (X, Y^2, Z^3)R.$$

Use this to find the irredundant parametric decomposition of $(X^6, Y^4, X^2Z^2, Z^6)R$.

Challenge Exercise 2. Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Let J be a monomial ideal in R. Prove or disprove: $\mathrm{C}_R(J^{[k]}) = \{z^{(k)} \mid z \in \mathrm{C}_R(J)$. (Maybe start with the cases $d = 1, 2$ first.)

Exercise 3. Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X, Y, Z]$ in three variables. Let $(\mathbf{X}) = (X, Y, Z)R$.

(a) Find an irredundant monomial generating sequence (in lexicographic order) for $\mathrm{P}_R(X^3Y^6Z^5) \cap \mathrm{P}_R(Y^3Z^6) \cap \mathrm{P}_R(X^4Y)$.

(b) Find an irredundant monomial generating sequence (in lexicographic order) for an ideal I such that $\mathrm{rad}\,(I) = \mathrm{rad}\,((\mathbf{X}))$ and $\mathrm{C}_R(I) = \{X^2Z, XYZ\}$.

(c) Is your answer for part (b) unique? Justify your response.

(d) Can you find a monomial ideal J such that $\mathrm{C}_R(J) = \{XY, YZ, XYZ\}$? Justify your response.

Exercise 4. Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables.

(a) Let $I_1, \ldots, I_k$, and J be monomial ideals in R. Prove that $(I_1 + \cdots + I_k) \cap J = (I_1 \cap J) + \cdots + (I_k \cap J)$.

(b) Give an example (where $d = 2 = k$) to show that this is not true without the assumption that each of the ideals I_1, I_2, and J are monomial ideals.