MONOMIAL IDEALS: HOMEWORK 9

Exercise 1. Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Set $(\mathbf{X}) = (X_1, \ldots, X_d)R$. Let I be a monomial ideal in R.

(a) Assume that $\operatorname{rad}(I) = \operatorname{rad}((\mathbf{X}))$. Prove that I has a unique monomial cover if and only if $I = \mathrm{P}_R(z)$ for some monomial $z \in [R]$.

(b) Prove that one implication in part (a) is still true when we drop the assumption that $\operatorname{rad}(I) = \operatorname{rad}((\mathbf{X}))$.

(c) Give an example to show that one implication in part (a) is false when we drop the assumption that $\operatorname{rad}(I) = \operatorname{rad}((\mathbf{X}))$.

Exercise 2. Construct an example to show that Proposition IV.1.13 is false without the assumption that $\operatorname{rad}(I) = \operatorname{rad}((\mathbf{X}))$, even if we assume that I has a monomial cover: Find monomial ideals I and Q such that $I \subsetneq Q$ and I has a monomial cover, but Q contains no monomial cover of I.

Exercise 3. Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X, Y, Z]$ in three variables.

(a) Set $I = (X^2, Y^3, Z^4)R$. Prove that I has a finite saturated chains of monomial ideals from I to R and that every such chain has the same length. What is the length?

(b) Repeat part (a) for the ideal $J = (X^2, Y^3, Z^4, XY^2Z^3)R$.

Definition 4. Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Let I and J be monomial ideals in R such that $I \subseteq J$. A *finite saturated chain of monomial ideals from I to J* is a finite chain of monomial ideals

$$I = I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_{n-1} \subsetneq I_n = J$$

such that I_j is a monomial cover of I_{j-1} for $1 \leqslant j \leqslant n$. The *length* of such a chain is the number of containments, in this case, n.

Exercise 5. Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Let $(\mathbf{X}) = (X_1, \ldots, X_d)R$, and let I and J be monomial ideals in R such that $I \subseteq J$.

(a) Assume that $J \neq R$. Prove that, if there exists a finite saturated chain of monomial ideals from I to J, then $\operatorname{rad}(I) = \operatorname{rad}(J)$.

(b) Prove that, if $\operatorname{rad}(I) = \operatorname{rad}((\mathbf{X}))$, then there exists a finite saturated chain of monomial ideals from I to J.

(c) Does the converse of part (a) hold? Justify your answer.