MATH 721, Algebra II
Exercises 1
Due Fri 18 Jan

Exercise 1. Let k be a field and let $f \in k[x]$ be a polynomial of degree 3.

(a) Show that either f has a root in k or is irreducible.
(b) Is the polynomial $x^3 - 5x^2 + 1$ irreducible over $\mathbb{Q}$?

Exercise 2. Show that $x^4 + 1$ is irreducible over $\mathbb{Q}$.

Exercise 3. Let R be a ring, $I \subset R$ a two-sided ideal and M a left R-module. Show that, if $am = 0$ for all $a \in I$ and all $m \in M$, then M has a well-defined R/I-module structure given by $(r + I)m = rm$. (Note that this shows that the class of R/I-modules is the same as the class of R-modules M such that $am = 0$ for all $a \in I$ and all $m \in M$.)

Exercise 4. Let R be a commutative ring and let M and N be R-modules.

(a) For each $f \in \mathrm{Hom}_R(M, N)$ and each $r \in R$, define $rf \colon M \to N$ by the formula $(rf)(m) := r(f(m)) = f(rm)$. Show that this definition makes $\mathrm{Hom}_R(M, N)$ into an R-module.
(b) Show that the natural bijection $\mathrm{Hom}_R(R^n, R^m) \to M_{m,n}(R)$ is an R-module isomorphism.