MATH 721, Algebra II
Exercises 2
Due Fri 25 Jan

Exercise 1. Let R be a ring, $I \subset R$ a left ideal and M a left R-module.

(a) Show that, if $\emptyset \neq S \subseteq M$, then $IS = \{\sum_{i=1}^{n} a_i s_i \mid n \in \mathbb{N}, a_i \in I, s_i \in S\}$ is a submodule of M.

(b) Assume that I is a two-sided ideal. Show that the R-module M/IM has a well-defined R/I-module structure given by $(r+I)(m+IM) = rm + IM$.

Exercise 2. Let R be a ring with identity. Show that every unital cyclic R-module is isomorphic to an R-module of the form R/J where J is a left ideal of R.

Exercise 3. Let R be a ring with identity. A unital R-module $M \neq 0$ is *simple* if its only submodules are 0 and M.

(a) Show that every simple R-module is cyclic.

(b) Assume that M is simple and $f\colon M \to M$ is an R-module homomorphism. Show that either $f = 0$ or f is an isomorphism.

(c) Assume that R is commutative. Show that M is simple if and only if $M \cong R/\mathfrak{m}$ where $\mathfrak{m}$ is a maximal ideal of R.

Exercise 4. Let R be a ring and let $f\colon M \to N$ and $g\colon N \to M$ be R-module homomorphisms such that $gf = \mathrm{id}_M$. Show that $N \cong \mathrm{Im}(f) \oplus \mathrm{Ker}(g)$.