MATH 721, Algebra II
Exercises 7
Due Fri 29 Feb

**Exercise 1.** Let $\phi\colon K \to L$ be a field extension. Assume that $L$ is algebraically closed, and let $F = \{u \in L \mid u \text{ is algebraic over } K\}$. Show that $F$ is an algebraic closure of $K$.

**Exercise 2.** Let $K$ be a finite field. Show that $K$ is not algebraically closed. [Hint: Let $K = \{a_1, \ldots, a_n\}$ with $a_1 \neq 0$ and consider $f = a_1 + (x - a_1) \cdots (x - a_n) \in K[x]$.]

**Exercise 3.** Let $\phi\colon K \to L$ be a field extension. Show that the following conditions are equivalent:

(i) $\phi\colon K \to L$ is an algebraic closure;
(ii) $\phi\colon K \to L$ is algebraic and, for every algebraic extension $\psi\colon K \to F$ there is a $K$-homomorphism $\gamma\colon F \to L$.

**Exercise 4.** Let $\alpha \in \mathbb{C}$ be a root of the polynomial $x^4 + 1$. Show that the only intermediate fields of the extension $\mathbb{Q} \subseteq \mathbb{Q}(\alpha)$ are the following: $\mathbb{Q}$, $\mathbb{Q}(\alpha)$, $\mathbb{Q}(\alpha^3 + \alpha)$, $\mathbb{Q}(\alpha^3 - \alpha)$ and $\mathbb{Q}(\alpha^2)$.