MATH 721, Algebra II
Exercise 14
Due Fri 25 April

Exercise 1. Let R be a commutative ring with identity and let I be a left ideal of R. For each unital R-module M, show that there is an isomorphism of R-modules

$$\mathrm{Hom}_R(R/I, M) \cong \{m \in M \mid Im = 0\}.$$

Exercise 2. Let R be a ring, and let P, Q be R-modules. Show that $P \oplus Q$ is projective if and only if P and Q are both projective.

Exercise 3. Let R be a ring and consider the following conditions:

(i) R is a field;
(ii) Every R-module is projective;
(iii) Every short exact sequence of R-modules is split.

Show that the implications (i) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (iii) always hold. Show that the conditions (i)–(iii) are equivalent when R is an integral domain. Does the implication (ii) $\Longrightarrow$ (i) hold in general?