Midterm 2 review
Math 165
Spring 2009

§3.7. Rates of Change in the Natural and Social Sciences.

If two quantities s and t are related by the equation $s = f(t)$, then the *instantaneous rate of change* of s with respect to t is $s' = f'(t) = \frac{ds}{dt}$.

For example, if a particle moves in a straight line path with distance from the origin $s = f(t)$ at time t, then the *instantaneous velocity* at time t is $v = f'(t)$, and the *instantaneous acceleration* at time t is $a = v' = f''(t)$.

Given a straight piece of wire where the mass of the part of the wire between the points x_0 and x is given as $m = f(x)$, then the *linear density* of the wire at a point x is described as $\rho(x) = f'(x)$.

§3.8. Exponential Growth and Decay.

Let $P = P(t)$ denote the population of a group of critters at time t. If the critters reproduce periodically and without constraint, then P satisfies the equation $\frac{dP}{dt} = kP$ for some number k. This means that $P(t) = P_0 e^{kt}$ where $P_0 = P(0)$.

Similarly, radioactive substances decay according to the same principle. Let $m = m(t)$ denote the mass of the substance at time t. Then $\frac{dm}{dt} = km$ for some number k, and so $m(t) = m_0 e^{kt}$ where $m_0 = m(0)$. The *half-life* t_h of the substance is the amount of time that it takes to reduce its mass to half the original mass: $m(t_h) = \frac{1}{2} m_0$.

Other quantities (like the temperature of a cooling object or the amount of money in a bank account) satisfy similar rules.

§3.9. Related Rates.

If two quantities a and b are related to each other by an equation, then their rates of change da/dt and db/dt are also related. Related rates problems deal with two (or more) quantities that are related in a specific way. You are then given information about all but one of the rates of change, and asked to determine the remaining rate of change, or to deduce some information given by the rate of change.

The key steps for related rates problems are:

1. Write an equation relating the given quantities according to the statement of the problem.

2. Differentiate both sides of the equation with respect to t.

3. Use the rates that you are given to solve for the unknown rate.

(See p. 243 for more details on the method.)

§3.10. Linear Approximations and Differentials.

Linear approximations are based on the following: $f(x) \approx f(a)+f'(a)(x-a)$. In other words, we have $f(a+\Delta x) \approx f(a)+f'(a)\Delta x$. This is used to approximate the quantity $f(x)$ when you know $f(a)$ and x is close to a. This principle can be used to estimate the error in the quantity $f(x)$ when there may be an error in the measurement of x.

§3.11. Hyperbolic Functions.

Hyperbolic functions are defined in terms of e^x and e^{-x}, and the inverse hyperbolic functions are defined as the corresponding inverse functions. Their definitions and derivatives are as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2} \qquad \frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$$

$$\frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}} \qquad \frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{x^2+1}}$$

§4.1. Maximum and Minimum Values.

Extreme Value Theorem: If f is continuous on an interval $[a, b]$, then f has an absolute minimum and an absolute maximum on that interval.

Fermat's Theorem: If f has a local maximum or minimum at c and $f'(c)$ exists, then $f'(c) = 0$.

A number c is a *critical number* for f if $f(c)$ exists and either $f'(c) = 0$ or $f'(c)$ does not exist.

Closed Interval Method: How to find the absolute minimum and an absolute maximum of a continuous function f on an interval $[a, b]$.
Step 1: Find each critical value c and compute $f(c)$.
Step 2: Compute $f(a)$ and $f(b)$.
Step 3: The largest value from Steps 1 and 2 is the absolute maximum. The smallest value from Steps 1 and 2 is the absolute minimum.

§4.2. The Mean Value Theorem.

Rolle's Theorem: Let f be a function such that
(1) f is continuous on the interval $[a, b]$,
(2) f is differentiable on the interval (a, b), and
(3) $f(a) = f(b)$.
Then there is a number c such that $a < c < b$ and $f'(c) = 0$.

This can be used, for instance, to show that equations have solutions.

Mean Value Theorem: Let f be a function such that
(1) f is continuous on the interval $[a, b]$, and
(2) f is differentiable on the interval (a, b).
Then there is a number c such that $a < c < b$ and $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

From this, we deduce the following:

Theorem: If $f'(x) = 0$ for all x in the interval (a, b), then there is a constant C such that $f(x) = C$ for all x in (a, b).

Corollary: If $f'(x) = g'(x)$ for all x in the interval (a, b), then there is a constant C such that $f(x) = g(x) + C$ for all x in (a, b).

These properties can be used to verify identities.

§4.3. How Derivatives Affect the Shape of a Graph.

Increasing/Decreasing Test:
(a) If $f'(x) > 0$ for all x in an interval I, then f is increasing on I.
(b) If $f'(x) < 0$ for all x in an interval I, then f is decreasing on I.

Here is how to find the intervals where f is increasing or decreasing:
Step 1: Find the critical values for f.
Step 2: Use test points to decide whether f' is positive or negative on each open interval determined by the critical values. These are the intervals where f is increasing or decreasing.

First Derivative Test: Assume that f is continuous and that c is a critical value for f.
(a) If f' changes from positive to negative at c, then f has a local max at c.
(b) If f' changes from negative to positive at c, then f has a local min at c.
(c) If f' does not change sign at c, then f has no local max or min at c.

Concavity Test:
(a) If $f''(x) > 0$ for all x in an interval I, then the graph of f is concave upward on I.
(b) If $f''(x) < 0$ for all x in an interval I, then the graph of f is concave downward on I.

A point P on a curve $y = f(x)$ is an *inflection point* if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Second Derivative Test: Assume that f'' is continuous near c.
(a) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local max at c.
(b) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local min at c.

For practice, work through the following:
homework assignments;
the even-numbered exercises that correspond to the homework exercises;
your old quizzes;
pp. 262–263 # 43, 45, 47, 48, 88, 89, 91–101, 102(a), 103(a), 105;
pp. 348–350 # 1–6, 15–18, 19–34 (as in Section 4.3), 45–48;
the practice exam

Practice exam.

1. Find the instantaneous rate of change of the area of the square with respect to side-length when the side-length is 10 cm.

2. A population of tribbles is initially 20 and grows at a rate that is proportional to its size. After 10 days, the population has increased to 30.
(a) How many tribbles will there be after 47 days?
(b) How long will it take for the population to reach 100?

3. Starting with a right triangle with sides 3,4,5 cm we increase the small side at a rate of 2 cm, keeping the side of length 4 constant. How fast is the smallest angle changing when it reaches the value of $\pi/4$ radians?

4. Use a linear approximation to estimate $\sqrt{4.01}$.

5. Find the absolute maximum value for the function $f(x) = x^3 - 3x + 1$ on the interval $[0, 2]$.

6. Show that the equation $x^7 + 2x^3 + 1 = 0$ has a solution in the interval $[-2, 1]$

7. Consider the function $f(x) = \dfrac{2x^2 - 1}{x^2 + 1}$.
(a) Find the critical values of f.
(b) Find the intervals where f is increasing and where f is decreasing.
(c) Find the local maxima and local minima for f.
(d) Find the intervals where the graph of f is concave upward and where it is concave downward.
(e) Find the inflection points for f.
(f) Find the asymptotes for the graph of f.
(g) Find the x-intercepts and the y-intercepts for the graph of f.
(h) Sketch the graph of f indicating the relevant data from (a)–(g).

Bonus. Find numbers b and c so that the point $(1, 0)$ is an inflection point for the function $x^3 + bx^2 + cx + 1$.