Final exam review
Math 165
Spring 2009

For review of Sections 2.1–4.3, see the review sheets for the midterms.

§4.4. Indeterminate Forms and L'Hospital's Rule.

L'Hospital's Rule: Assume that f and g are differentiable and that $g'(x) \neq 0$ on an open interval that contains a (except possibly at a). Assume that the limit $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ is of the form $0/0$ or $\pm\infty/\pm\infty$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right exists (or is $\pm\infty$). This also works for limits at $\pm\infty$.

This can be used to compute $\lim_{x\to a} f(x)g(x)$ of the form $\pm\infty \cdot 0$ by rewriting

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} \frac{f(x)}{1/g(x)} \quad \text{or} \quad \lim_{x\to a} \frac{g(x)}{1/f(x)}.$$

It can also be used to compute $\lim_{x\to a}[f(x) - g(x)]$ of the form $\infty - \infty$ by rewriting $f(x) - g(x)$ with a common denominator.

It can also be used to compute $L = \lim_{x\to a} f(x)^{g(x)}$ of the form 0^0 or ∞^0 or 1^∞ by taking a logarithm:

$$\ln(L) = \ln(\lim_{x\to a} f(x)^{g(x)}) = \lim_{x\to a} \ln(f(x)^{g(x)}) = \lim_{x\to a}[g(x)\ln(f(x))].$$

§4.7. Optimization Problems.

Exercises from this section are "word problems" that ask us to use calculus to maximize or minimize some quantity. Here is a summary of the steps from p. 322 of the text.

1. Take inventory. What is given? What is being asked for?

2. Draw a diagram.

3. Introduce notation. Say that Q denotes the quantity that you are trying to maximize or minimize. This depends on other quantities which you can call a, b, c, x, y, z, θ, etc.

4. Find an equation that describes a relation between Q and the other variables.

5. Use the given information for write Q in terms of one variable.

6. Use the methods of Sections 4.1 and/or 4.3 to find the absolute maximum or absolute minimum for Q. Be sure to answer the question that was asked.

§4.9. Antiderivatives.

Antidifferentiation is the reverse process of differentiation. A function F is an antiderivative of f provided that $F'(x) = f(x)$. In this case, the general antiderivative of f is $F(x) + C$, where C is an arbitrary constant.

§5.1. Area and Distances.

Be able to approximate the area under a curve by using rectangles.

Be able to approximate distance traveled by using velocity readings and small time intervals.

Be able to express area (or distance traveled) as a limit.

§5.2. The Definite Integral.

Let f be defined on an interval $[a, b]$ and let n be a positive integer. Set $\Delta x = (b - a)/n$, and write $x_i = a + i\Delta x$ for $i = 0, 1, \ldots, n$. The definite integral of f is the limit

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i^*)\Delta x\right]$$

where x_i^* is in the interval $x_{i-1}, x_i]$.

Be able to compute a definite integral using the limit definition.

Be able to use the following properties of integrals:

1. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.
2. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$.
3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.
4. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$.
5. $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

6. $\int_a^a f(x)\,dx = 0$.

7. If $f(x) \geq 0$ and $a \leq b$, then $\int_a^b f(x)\,dx \geq 0$.

8. If $f(x) \geq g(x)$ and $a \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

9. If $m \leq f(x) \leq M$ and $a \leq b$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

Be able to compute integrals by interpreting them as areas:

If $f(x) \geq 0$ and $a \leq b$, then $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ above the interval $[a, b]$.

If $f(x) \leq 0$ and $a \leq b$, then $-\int_a^b f(x)\,dx$ is the area above the curve $y = f(x)$ under the interval $[a, b]$.

§5.3. The Fundamental Theorem of Calculus.

The two forms of the Fundamental Theorem of Calculus show that integration is the inverse operation of differentiation.

Fundamental Theorem of Calculus 1: If f is continuous on the interval $[a, b]$, then the function g defined as

$$g(x) = \int_a^x g(t)\,dt \qquad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

Fundamental Theorem of Calculus 2: Assume that f is continuous and that F is an antiderivative of f. Then

$$\int_a^b f(x)\,dx = F(x)\big]_a^b = F(b) - F(a).$$

§5.4. Indefinite Integrals and the Net Change Theorem.

The notation for the general antiderivative of a function f is

$$\int f(x)\,dx = F(x) + C.$$

See p. 392 of the text for a list of antiderivatives to memorize.

§5.5. The Substitution Rule.

The Substitution Rule is the reverse of the Chain Rule. If $u = g(x)$ and $du = g'(x)dx$, then

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du.$$

This process is called "u-substitution".

For definite integrals, we have

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

For practice, work through the following:
homework assignments;
the even-numbered exercises (and other odd-numbered exercises) that correspond to the homework exercises;
your old quizzes;
pp. 348–350, Exercises # 7–14, 50–59, 65–7477–79;
pp. 409–411, Exercises # 1–5, 8–38, 43–56, 60, 61, 65–68;
the practice exam

Practice exam.

1. Compute the following limits:

(a) $\lim_{x \to -1} \dfrac{x^2 + x}{e^x - 1}$

(b) $\lim_{x \to 0} \dfrac{x^2 + x}{e^x - 1}$

2. Compute the derivatives of the following functions:

(a) $f(x) = \ln(e^x + 1)$

(b) $g(x) = \dfrac{\sqrt{x+2}}{\tan x - 3}$

(c) $h(x) = (x^2 + 1)^{(x^2+1)}$

3. Compute the derivative of $f(x) = x^2 + x$, using the definition of the derivative.

4. Find a point where the tangent line to the curve $y = \cos x$ has slope $1/2$.

5. The volume of a cube is increasing at a rate of $10\,\text{cm}^3/\text{min}$. How fast is the surface area increasing when the length of an edge is 30 cm?

6. Find the relative maximum and relative minimum points of the function $f(x) = x^4 - x^3 + x^2 - 1$.

7. A cone-shaped drinking cup is to be made to hold $27\,\text{cm}^3$ of water. Find the height and radius of the cup that will use the smallest amount of paper.

8. Compute the following integrals:

(a) $\int_1^3 (x^2 + x)\,dx$

(b) $\int (\sqrt{x^3} + 1)^2\,dx$

(c) $\int \sin(\sqrt{x^3} + 1)\sqrt{x}\,dx$

(d) $\int \dfrac{1}{x \ln x}\,dx$

9. Find a number b such that the area under the curve $y = x/(1+x^2)$ above the interval $[0, b]$ is 1.

Bonus. Compute $\int_1^3 (x^2 + x)\,dx$ using the definition of the definite integral.