Paths to understanding birational rowmotion

Gregg Musiker (UMN)* and Tom Roby (UCONN)**

Special Session on Dynamical Algebraic Combinatorics
Joint Mathematics Meeting of the AMS and MAA, San Diego

13 January 2018

arXiv:1801.03877v1

Outline

1. Classical rowmotion
2. Birational rowmotion
3. Formula for birational rowmotion in terms of Lattice Paths
4. Sketch of proof
5. Further applications

Thank you for support from NSF Grant DMS-1362980 and
the 2015 AIM workshop on Dynamical Algebraic Combinatorics.

arXiv:1801.03877v1

Classical rowmotion is the rowmotion studied by Striker-Williams (`arXiv:1108.1172`). It has appeared many times before, under different guises:

- Brouwer-Schrijver (1974) (as a permutation of the antichains),
- Fon-der-Flaass (1993) (as a permutation of the antichains),
- Cameron-Fon-der-Flaass (1995) (as a permutation of the monotone Boolean functions),
- Panyushev (2008), Armstrong-Stump-Thomas (2011) (as a permutation of the antichains or "nonnesting partitions", with relations to Lie theory).
- Several times before in this special session! (So I give it short shrift.)

Motivations and Connections

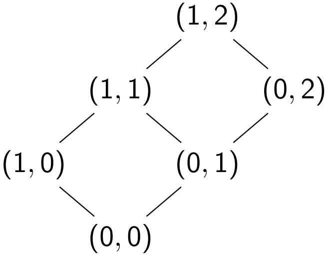
- Classical rowmotion is closely related to the Auslander-Reiten translation in quivers arising in certain special posets (e.g., rectangles) [Yil17].
- Birational rowmotion can be related to Y-systems of type $A_m \times A_n$ described in Zamolodchikov periodicity [Rob16, §4.4].
- The orbits of these actions all have natural *homomesic* statistics [PR13, EiPr13, EiPr14].
- Periodicity of these systems is generally nontrivial to prove.

Classical rowmotion: Periodicity

Classical rowmotion is a permutation of $J(P)$, hence has finite order. This order can be fairly large.

Classical rowmotion is a permutation of $J(P)$, hence has finite order. This order can be fairly large.
However, **for some types of** P, the order can be explicitly computed or bounded from above. See Striker-Williams [StWi11] (and the **very recent** Thomas-Williams [TW17]) for an exposition of known results.

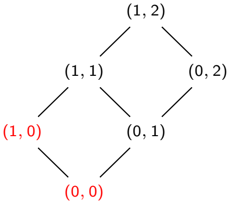
- If P is a $[0, r] \times [0, s]$-rectangle:



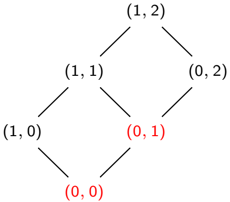
(shown here for $r = 1$ and $s = 2$), then $\operatorname{ord}(\mathbf{r}) = r + s + 2 = 5$.

Classical rowmotion: Periodicity (Example)

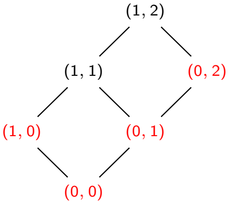
$S =$



, $\mathbf{r}(S) =$

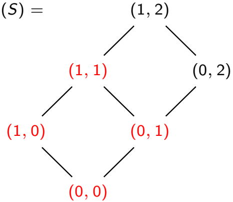


, $\mathbf{r}^2(S) =$

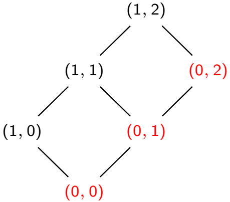


,

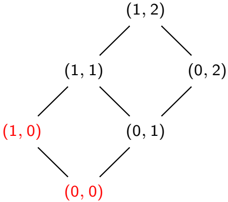
$\mathbf{r}^3(S) =$



, $\mathbf{r}^4(S) =$

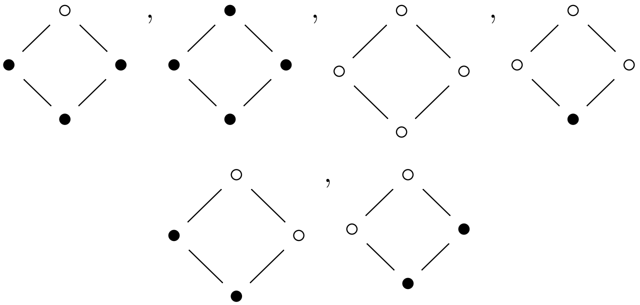


, $\mathbf{r}^5(S) =$



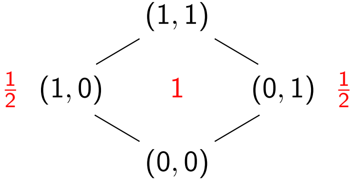
,

which is precisely the S we started with. $\text{ord}(\mathbf{r}) = p + q = 2 + 3 = 5.$



The average value along antipodal (N-S, E-W) pairs is 1 for both orbits,

and is also **constant**, as



, on files (columns).

There is an alternative definition of rowmotion, which splits it into many small operations, each an involution.

- Define $\mathbf{t}_v(S)$ as:
  - $S \triangle \{v\}$ (symmetric difference) if this is an order ideal;
  - S otherwise.

  ("Try to add or remove v from S, as long as the result remains an order ideal, i.e., within $J(P)$; otherwise, leave S fixed.")

There is an alternative definition of rowmotion, which splits it into many small operations, each an involution.

- Define $\mathbf{t}_v(S)$ as:
  - $S \triangle \{v\}$ (symmetric difference) if this is an order ideal;
  - S otherwise.

  ("Try to add or remove v from S, as long as the result remains an order ideal, i.e., within $J(P)$; otherwise, leave S fixed.")
- Note that $\mathbf{t}_v^2 = \mathrm{id}$.
- Let $(v_1, v_2, ..., v_n)$ be a **linear extension** of P; this means a list of all elements of P (each only once) such that $i < j$ whenever $v_i < v_j$.
- Cameron and Fon-der-Flaass [CaFl95] showed that

$$\mathbf{r} = \mathbf{t}_{v_1} \circ \mathbf{t}_{v_2} \circ ... \circ \mathbf{t}_{v_n}.$$

Generalizing to the piecewise-linear setting

The decomposition of classical rowmotion into toggles allows us to define a **piecewise-linear (PL)** version of rowmotion acting on functions on a poset.

The decomposition of classical rowmotion into toggles allows us to define a **piecewise-linear (PL)** version of rowmotion acting on functions on a poset.

Let P be a poset, with an extra minimal element $\hat{0}$ and an extra maximal element $\hat{1}$ adjoined.

Generalizing to the piecewise-linear setting

The decomposition of classical rowmotion into toggles allows us to define a **piecewise-linear (PL)** version of rowmotion acting on functions on a poset.

Let P be a poset, with an extra minimal element $\hat{0}$ and an extra maximal element $\hat{1}$ adjoined.

The **order polytope** $\mathcal{O}(P)$ (introduced by R. Stanley) is the set of functions $f : P \to [0, 1]$ with $f(\hat{0}) = 0$, $f(\hat{1}) = 1$, and $f(x) \leq f(y)$ whenever $x \leq_P y$.

Generalizing to the piecewise-linear setting

The decomposition of classical rowmotion into toggles allows us to define a **piecewise-linear (PL)** version of rowmotion acting on functions on a poset.

Let P be a poset, with an extra minimal element $\hat{0}$ and an extra maximal element $\hat{1}$ adjoined.

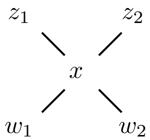
The **order polytope** $\mathcal{O}(P)$ (introduced by R. Stanley) is the set of functions $f : P \to [0,1]$ with $f(\hat{0}) = 0$, $f(\hat{1}) = 1$, and $f(x) \leq f(y)$ whenever $x \leq_P y$.

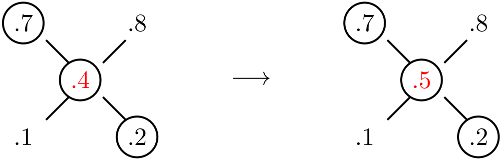
For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending f to the unique f' satisfying

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x) & \text{if } y = x, \end{cases}$$

where $z \cdot > x$ means z covers x and $w < \cdot x$ means x covers w.

Example of flipping at a node



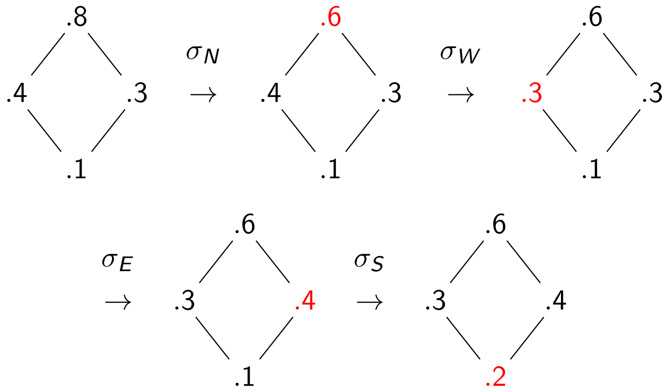


$$\min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) = .7 + .2 = .9$$

$$f(x) + f'(x) = .4 + .5 = .9$$

Composing flips

Just as we can apply toggle-maps from top to bottom, we can apply flip-maps from top to bottom, to get *piecewise-linear rowmotion*:



(We successively flip at $N = (1,1)$, $W = (1,0)$, $E = (0,1)$, and $S = (0,0)$ in order.)

De-tropicalizing to birational maps

In the so-called *tropical semiring*, one replaces the standard binary ring operations $(+, \cdot)$ with the tropical operations $(\max, +)$. In the piecewise-linear (PL) category of the order polytope studied above, our flipping-map at x replaced the value of a function $f : P \to [0, 1]$ at a point $x \in P$ with f', where

$$f'(x) := \min_{z \cdot > x} f(z) + \max_{w < \cdot x} f(w) - f(x)$$

We can "detropicalize" this flip map and apply it to an assignment $f : P \to \mathbb{R}(x)$ of *rational functions* to the nodes of the poset, using that $\min(z_i) = -\max(-z_i)$, to get the **birational toggle map**

$$(T_x f)(x) = f'(x) = \frac{\sum_{w < \cdot x} f(w)}{f(x) \sum_{z \cdot > x} \frac{1}{f(z)}}$$

- Let P be a finite poset. We define $\widehat{P}$ to be the poset obtained by adjoining two new elements $\widehat{0}$ and $\widehat{1}$ to P and forcing
  - $\widehat{0}$ to be less than every other element, and
  - $\widehat{1}$ to be greater than every other element.
- Let $\mathbb{K}$ be a field.
- A **$\mathbb{K}$-labelling of** P will mean a function $f : \widehat{P} \to \mathbb{K}$.
- We will represent labellings by drawing the labels on the vertices of the Hasse diagram of $\widehat{P}$.
- For any $v \in P$, define the **birational v-toggle** as the rational map $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ by $(T_v f)(w) = \frac{\sum_{\widehat{P} \ni u \lessdot v} f(u)}{f(v) \sum_{\widehat{P} \ni u \gtrdot v} \frac{1}{f(u)}}$ for $w = v$.

  (We leave $(T_v f)(w) = f(w)$ when $w \neq v$.)

- For any $v \in P$, define the **birational v-toggle** $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ by $(T_v f)(w) = \frac{\sum_{u \lessdot v} f(u)}{f(v) \sum_{u \gtrdot v} \frac{1}{f(u)}}$ for $w = v$.
- Notice that this is a **local change** only to the label at v.
- We have $T_v^2 = id$ (on the range of T_v), and T_v is a birational map.

- For any $v \in P$, define the **birational v-toggle**

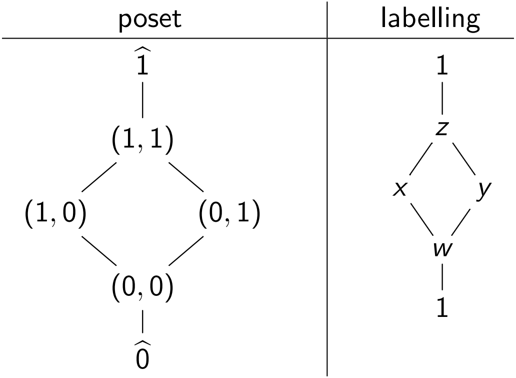
  $T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ by $(T_v f)(w) = \frac{\sum_{u \lessdot v} f(u)}{f(v) \sum_{u \gtrdot v} \frac{1}{f(u)}}$ for $w = v$.
- Notice that this is a **local change** only to the label at v.
- We have $T_v^2 = id$ (on the range of T_v), and T_v is a birational map.
- We define **birational rowmotion** as the rational map

  $$\rho_B := T_{v_1} \circ T_{v_2} \circ \ldots \circ T_{v_n} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}},$$

  where $(v_1, v_2, \ldots, v_n)$ is a linear extension of P.
- This is indeed independent of the linear extension, because
  - T_v and T_w commute whenever v and w are incomparable (even whenever they are not adjacent in the Hasse diagram of P);
  - we can get from any linear extension to any other by switching incomparable adjacent elements.
- This is originally due to Einstein and Propp [EiPr13, EiPr14]. Another exposition of these ideas can be found in [Rob16], from the IMA volume *Recent Trends in Combinatorics*.

Example:

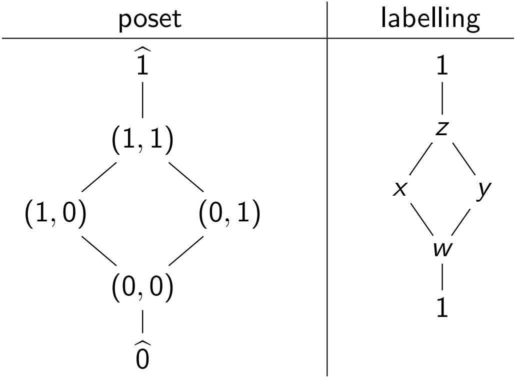
Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:



Birational rowmotion: example

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

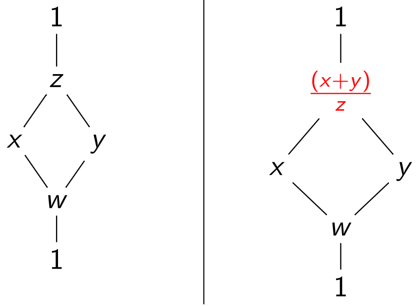


We have $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$
using the linear extension $((1,1),(1,0),(0,1),(0,0))$.

That is, toggle in the order "top, left, right, bottom".

Example:
Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

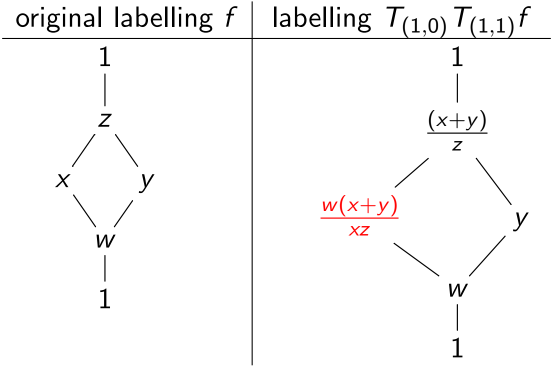
original labelling f	labelling $T_{(1,1)}f$



We are using $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$.

Example:

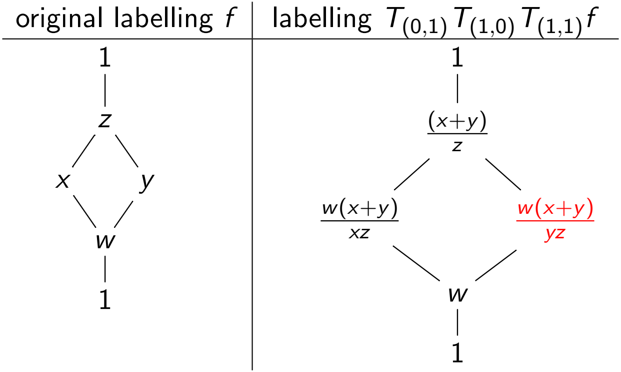
Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:



We are using $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$.

Example:

Let us “rowmote” a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

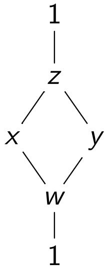


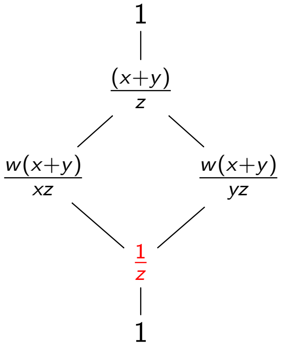
We are using $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$.

Example:

Let us “rowmote” a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(0,0)} T_{(0,1)} T_{(1,0)} T_{(1,1)} f = \rho_B f$





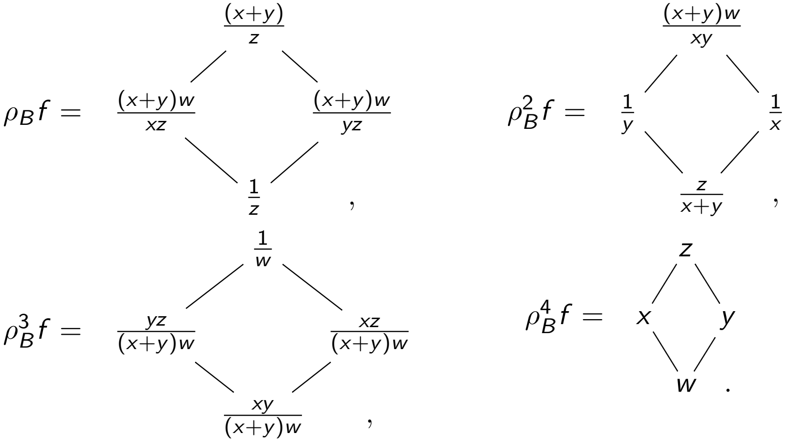
We are using $\rho_B = {\color{red}T_{(0,0)}} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$.

Example: Iterating this procedure we get

$$\rho_B f = \begin{array}{ccc} & \frac{(x+y)}{z} & \\ \frac{(x+y)w}{xz} & & \frac{(x+y)w}{yz} \\ & \frac{1}{z} & \end{array}, \qquad \rho_B^2 f = \begin{array}{ccc} & \frac{(x+y)w}{xy} & \\ \frac{1}{y} & & \frac{1}{x} \\ & \frac{z}{x+y} & \end{array},$$

$$\rho_B^3 f = \begin{array}{ccc} & \frac{1}{w} & \\ \frac{yz}{(x+y)w} & & \frac{xz}{(x+y)w} \\ & \frac{xy}{(x+y)w} & \end{array}, \qquad \rho_B^4 f = \begin{array}{ccc} & z & \\ x & & y \\ & w & \end{array}.$$

Example: Iterating this procedure we get



Notice that $\rho_B^4 f = f$, which generalizes to $\rho_B^{r+s+2} f = f$ for $P = [0, r] \times [0, s]$ [Grinberg-R 2015]. Notice also "antipodal reciprocity".

Birational homomesy on files, (aka columns)

The poset $[0,1] \times [0,1]$ has three files, $\{(1,0)\}$, $\{(0,0),(1,1)\}$, and $\{(0,1)\}$. Multiplying over all iterates of birational rowmotion in a given file, we get

$$\rho_B(f)(1,0)\rho_B^2(f)(1,0)\rho_B^3(f)(1,0)\rho_B^4(f)(1,0) = \frac{(x+y)w}{xz} \ \frac{1}{y} \ \frac{yz}{(x+y)w} \ (x) = 1,$$

Birational homomesy on files, (aka columns)

The poset $[0,1] \times [0,1]$ has three files, $\{(1,0)\}$, $\{(0,0),(1,1)\}$, and $\{(0,1)\}$. Multiplying over all iterates of birational rowmotion in a given file, we get

$$\rho_B(f)(1,0)\rho_B^2(f)(1,0)\rho_B^3(f)(1,0)\rho_B^4(f)(1,0) = \frac{(x+y)w}{xz} \;\; \frac{1}{y} \;\; \frac{yz}{(x+y)w} \;\; (x) = 1,$$

$$\rho_B(f)(0,0)\rho_B(f)(1,1)\rho_B^2(f)(0,0)\rho_B^2(f)(1,1)\rho_B^3(f)(0,0)\rho_B^3(f)(1,1)\rho_B^4(f)(0,0)\rho_B^4(f)(1,1) =$$

$$\frac{1}{z} \;\; \frac{x+y}{z} \;\; \frac{z}{x+y} \;\; \frac{(x+y)w}{xy} \;\; \frac{xy}{(x+y)w} \;\; \frac{1}{w} \;\; (x) \;\; (z) = 1,$$

Birational homomesy on files, (aka columns)

The poset $[0,1] \times [0,1]$ has three files, $\{(1,0)\}$, $\{(0,0),(1,1)\}$, and $\{(0,1)\}$. Multiplying over all iterates of birational rowmotion in a given file, we get

$$\rho_B(f)(1,0)\rho_B^2(f)(1,0)\rho_B^3(f)(1,0)\rho_B^4(f)(1,0) = \frac{(x+y)w}{xz} \ \frac{1}{y} \ \frac{yz}{(x+y)w} \ (x) = 1,$$

$$\rho_B(f)(0,0)\rho_B(f)(1,1)\rho_B^2(f)(0,0)\rho_B^2(f)(1,1)\rho_B^3(f)(0,0)\rho_B^3(f)(1,1)\rho_B^4(f)(0,0)\rho_B^4(f)(1,1) =$$

$$\frac{1}{z} \ \frac{x+y}{z} \ \frac{z}{x+y} \ \frac{(x+y)w}{xy} \ \frac{xy}{(x+y)w} \ \frac{1}{w} \ (x) \ (z) = 1,$$

$$\rho_B(f)(0,1)\rho_B^2(f)(0,1)\rho_B^3(f)(0,1)\rho_B^4(f)(0,1) = \frac{(x+y)w}{yz} \ \frac{1}{x} \ \frac{xz}{(x+y)w} \ (y) = 1.$$

Birational homomesy on files, (aka columns)

The poset $[0,1] \times [0,1]$ has three files, $\{(1,0)\}$, $\{(0,0),(1,1)\}$, and $\{(0,1)\}$. Multiplying over all iterates of birational rowmotion in a given file, we get

$$\rho_B(f)(1,0)\rho_B^2(f)(1,0)\rho_B^3(f)(1,0)\rho_B^4(f)(1,0) = \frac{(x+y)w}{xz} \; \frac{1}{y} \; \frac{yz}{(x+y)w} \; (x) = 1,$$

$$\rho_B(f)(0,0)\rho_B(f)(1,1)\rho_B^2(f)(0,0)\rho_B^2(f)(1,1)\rho_B^3(f)(0,0)\rho_B^3(f)(1,1)\rho_B^4(f)(0,0)\rho_B^4(f)(1,1) =$$

$$\frac{1}{z} \; \frac{x+y}{z} \; \frac{z}{x+y} \; \frac{(x+y)w}{xy} \; \frac{xy}{(x+y)w} \; \frac{1}{w} \; (x) \; (z) = 1,$$

$$\rho_B(f)(0,1)\rho_B^2(f)(0,1)\rho_B^3(f)(0,1)\rho_B^4(f)(0,1) = \frac{(x+y)w}{yz} \; \frac{1}{x} \; \frac{xz}{(x+y)w} \; (y) = 1.$$

Each of these products equalling one is the manifestation, for the poset of a product of two chains, of **homomesy along files** at the birational level.

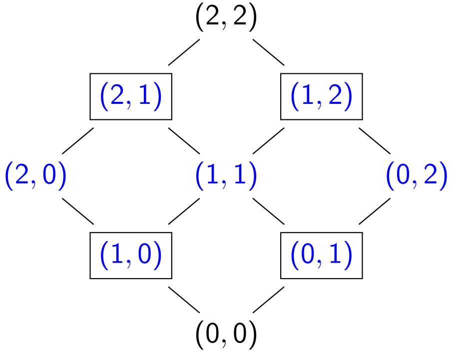
Birational Rowmotion on the Rectangular Poset

We now give a rational function formula for the values of iterated birational rowmotion $\rho_B^{k+1}(i,j)$ for $(i,j) \in [0,r] \times [0,s]$ and $k \in [0, r+s+1]$.

Birational Rowmotion on the Rectangular Poset

We now give a rational function formula for the values of iterated birational rowmotion $\rho_B^{k+1}(i,j)$ for $(i,j) \in [0,r] \times [0,s]$ and $k \in [0, r+s+1]$.

1) Let $\bigvee_{(m,n)} := \{(u,v) : (u,v) \geq (m,n)\}$ be the *principal order filter at* (m,n), $\hexagon^k_{(m,n)}$be the *rank-selected subposet*, of elements in $\bigvee_{(m,n)}$ whose rank (within $\bigvee_{(m,n)}$) is at least $k-1$ and whose corank is at most $k-1$.



2) Let $s_1, s_2, \ldots, s_k$ be the k minimal elements and let $t_1, t_2, \ldots, t_k$ be the k maximal elements of $\hexagon^k_{(m,n)}$.

2) Let $s_1, s_2, \ldots, s_k$ be the k minimal elements and let $t_1, t_2, \ldots, t_k$ be the k maximal elements of $\hexagon^k_{(m,n)}$.

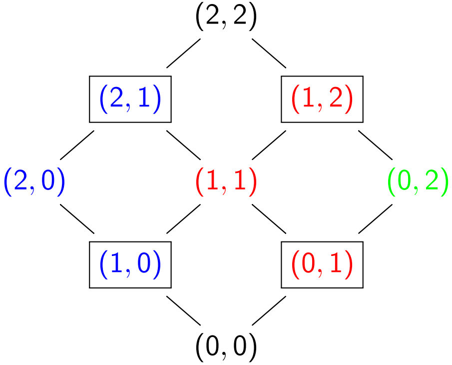
Let $A_{ij} := \frac{\sum_{z \lessdot (i,j)} x_z}{x_{(i,j)}} = \frac{x_{i,j-1} + x_{i-1,j}}{x_{ij}}$. We set $x_{i,j} = 0$ for $(i,j) \notin P$ and $A_{00} = \frac{1}{x_{00}}$ (working in $\widehat{P}$).

Given a triple $(k, m, n) \in \mathbb{N}^3$, we define a polynomial $\varphi_{\mathbf{k}}(\mathbf{m}, \mathbf{n})$ in terms of the A_{ij}'s as follows.

Birational Rowmotion on the Rectangular Poset

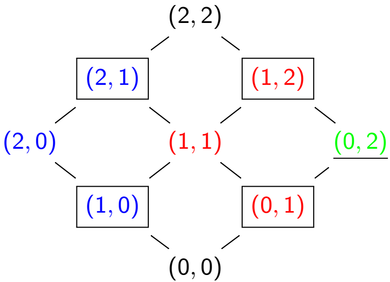
We define a **lattice path of length** ℓ within $P = [0, r] \times [0, s]$ to be a sequence $v_1, v_2, \ldots, v_\ell$ of elements of P such that each difference of successive elements $v_i - v_{i-1}$ is either $(1, 0)$ or $(0, 1)$ for each $i \in [\ell]$. We call a collection of lattice paths **non-intersecting** if no two of them share a common vertex.

EG: The blue path and red path below are non-intersecting.



3) Let $S_k(m,n)$ be the set of non-intersecting lattice paths in $\hexagon^k_{(m,n)}$, from $\{s_1, s_2, \ldots, s_k\}$ to $\{t_1, t_2, \ldots, t_k\}$. Let $\mathcal{L} = (L_1, L_2, \ldots L_k) \in S^k_k(m,n)$ denote a k-tuple of such lattice paths.

4) Define $\varphi_k(m,n) :=$
$\sum_{\mathcal{L} \in S^k_k(m,n)} \prod_{\substack{(i,j) \in \hexagon^k_{(m,n)} \\ (i,j) \notin L_1 \cup L_2 \cup \cdots \cup L_k}} A_{ij}.$



Theorem(*):

$$\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k, j-k)}{\varphi_{k+1}(i-k, j-k)}$$

EG: $\rho_B^2(1,1) = \frac{\varphi_1(0,0)}{\varphi_2(0,0)}$.

$$= \frac{\text{sum of 6 quartic terms in } A_{ij}}{A_{20} + A_{11} + A_{02}}$$

(*) Caveats explained and general statement given in the next few slides.

Main Theorem (Musiker-R 2018)

Fix $k \in [0, r+s+1]$, and let $\rho_B^{k+1}(i,j)$ denote the rational function associated to the poset element (i,j) after $(k+1)$ applications of the birational rowmotion map to the generic initial labeling of $P = [0,r] \times [0,s]$. Set $[\alpha]_+ := \max\{\alpha, 0\}$ and $M = [k-i]_+ + [k-j]_+$.

Main Theorem (Musiker-R 2018)

Fix $k \in [0, r+s+1]$, and let $\rho_B^{k+1}(i,j)$ denote the rational function associated to the poset element (i,j) after $(k+1)$ applications of the birational rowmotion map to the generic initial labeling of $P = [0,r] \times [0,s]$. Set $[\alpha]_+ := \max\{\alpha, 0\}$ and $M = [k-i]_+ + [k-j]_+$.

(a1) When $M = 0$, i.e., $(i-k, j-k)$ still lies in the poset $[0,r] \times [0,s]$:

$$\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k, j-k)}{\varphi_{k+1}(i-k, j-k)}$$

where $\varphi_t(v,w)$ is defined in 4) above.

Main Theorem (Musiker-R 2018)

Fix $k \in [0, r+s+1]$, and let $\rho_B^{k+1}(i,j)$ denote the rational function associated to the poset element (i,j) after $(k+1)$ applications of the birational rowmotion map to the generic initial labeling of $P = [0,r] \times [0,s]$. Set $[\alpha]_+ := \max\{\alpha, 0\}$ and $M = [k-i]_+ + [k-j]_+$.

(a1) When $M = 0$, i.e., $(i-k, j-k)$ still lies in the poset $[0,r] \times [0,s]$:

$$\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k, j-k)}{\varphi_{k+1}(i-k, j-k)}$$

where $\varphi_t(v,w)$ is defined in 4) above.
(a2) When $0 < M \leq k$:

$$\rho_B^{k+1}(i,j) = \mu^{([k-j]_+, [k-i]_+)} \left(\frac{\varphi_{k-M}(i-k+M, j-k+M)}{\varphi_{k-M+1}(i-k+M, j-k+M)} \right)$$

where $\mu^{(a,b)}$ is the operator that takes a rational function in $\{A_{(u,v)}\}$ and simply shifts each index in each factor of each term: $A_{(u,v)} \mapsto A_{(u-a,v-b)}$.

Main Theorem (Musker-R 2018)

Fix $k \in [0, r+s+1]$ and set $M = [k-i]_+ + [k-j]_+$. After $(k+1)$ applications of the birational rowmotion map to the generic initial labeling of $P = [0,r] \times [0,s]$ we get:

(a) When $0 \leq M \leq k$:

$$\rho_B^{k+1}(i,j) = \mu^{([k-j]_+,[k-i]_+)} \left(\frac{\varphi_{k-M}(i-k+M, j-k+M)}{\varphi_{k-M+1}(i-k+M, j-k+M)} \right)$$

where $\varphi_t(v,w)$ and $\mu^{(a,b)}$ are as defined above.

Main Theorem (Musker-R 2018)

Fix $k \in [0, r+s+1]$ and set $M = [k-i]_+ + [k-j]_+$. After $(k+1)$ applications of the birational rowmotion map to the generic initial labeling of $P = [0,r] \times [0,s]$ we get:

(a) When $0 \leq M \leq k$:

$$\rho_B^{k+1}(i,j) = \mu^{([k-j]_+,[k-i]_+)} \left(\frac{\varphi_{k-M}(i-k+M, j-k+M)}{\varphi_{k-M+1}(i-k+M, j-k+M)} \right)$$

where $\varphi_t(v,w)$ and $\mu^{(a,b)}$ are as defined above.

(b) When $M \geq k$: $\rho_B^{k+1}(i,j) = 1/\rho_B^{k-i-j}(r-i, s-j)$, which is well-defined by part (a).

Remark: We prove that our formulae in (a) and (b) agree when $M = k$, allowing us to give a new proof of periodicity: $\rho_B^{r+s+2+d} = \rho_B^d$; thus we get a formula for **all** iterations of the birational rowmotion map.

Corollary

For $k \le \min\{i,j\}$, $\rho_B^{k+1}(i,j) = \frac{\varphi_k(i-k,j-k)}{\varphi_{k+1}(i-k,j-k)}$.

Corollary ([GrRo15, Thm. 30, 32])

The birational rowmotion map ρ_B *on the product of two chains* $P = [0,r] \times [0,s]$ *is* **(1)** *periodic, with period* $r+s+2$, *and* **(2)** *satisfies antipodal reciprocity* $\rho_B^{i+j+1} = 1/\rho_B^0(r-i,s-j) = \frac{1}{x_{r-i,s-j}}$.

Theorem

Given a file F *in* $[0,r] \times [0,s]$, $\prod_{k=0}^{r+s+1} \prod_{(i,j)\in F} \rho_B^k(i,j) = 1$.

Example of Path Formula

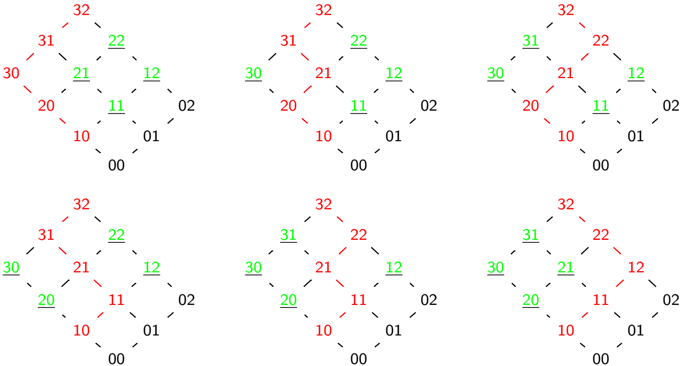
We use our main theorem to compute $\rho_B^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for
Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.
When k = 1, we still have $M = 0$, and $\rho_B^2(2,1) = \frac{\varphi_1(1,0)}{\varphi_2(1,0)} =$

$$\frac{A_{11}A_{12}A_{21}A_{22} + A_{11}A_{12}A_{22}A_{30} + A_{11}A_{12}A_{30}A_{31} + A_{12}A_{20}A_{22}A_{30} + A_{12}A_{20}A_{30}A_{31} + A_{20}A_{21}A_{30}A_{31}}{A_{12} + A_{21} + A_{30}}.$$

Example of Path Formula

We use our main theorem to compute $\rho_B^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for
Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.
When k = 1, we still have $M = 0$, and $\rho_B^2(2,1) = \frac{\varphi_1(1,0)}{\varphi_2(1,0)} =$

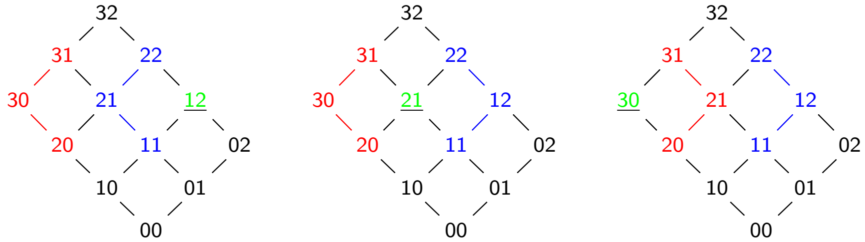
$$\frac{A_{11}A_{12}A_{21}A_{22} + A_{11}A_{12}A_{22}A_{30} + A_{11}A_{12}A_{30}A_{31} + A_{12}A_{20}A_{22}A_{30} + A_{12}A_{20}A_{30}A_{31} + A_{20}A_{21}A_{30}A_{31}}{A_{12} + A_{21} + A_{30}}.$$



Example of Path Formula

We use our main theorem to compute $\rho_B^{k+1}(2,1)$ for $P = [0,3] \times [0,2]$ for
Here $r = 3, s = 2, i = 2$, and $j = 1$ throughout.
When k = 1, we still have $M = 0$, and $\rho_B^2(2,1) = \frac{\varphi_1(1,0)}{\varphi_2(1,0)} =$

$$\frac{A_{11}A_{12}A_{21}A_{22} + A_{11}A_{12}A_{22}A_{30} + A_{11}A_{12}A_{30}A_{31} + A_{12}A_{20}A_{22}A_{30} + A_{12}A_{20}A_{30}A_{31} + A_{20}A_{21}A_{30}A_{31}}{A_{12} + A_{21} + A_{30}}.$$



Sketch of Proof

By the definition of birational rowmotion,

$$\rho_B^{k+1}(i,j) = \frac{\left(\rho_B^k(i,j-1) + \rho_B^k(i-1,j)\right) \cdot \left(\rho_B^{k+1}(i+1,j) \mid\mid \rho_B^{k+1}(i,j+1)\right)}{\rho_B^k(i,j)}$$

where

$$A \mid\mid B = \frac{1}{\frac{1}{A} + \frac{1}{B}}.$$

Sketch of Proof

By the definition of birational rowmotion,

$$\rho_B^{k+1}(i,j) = \frac{\left(\rho_B^k(i,j-1) + \rho_B^k(i-1,j)\right) \cdot \left(\rho_B^{k+1}(i+1,j) \parallel \rho_B^{k+1}(i,j+1)\right)}{\rho_B^k(i,j)}$$

where

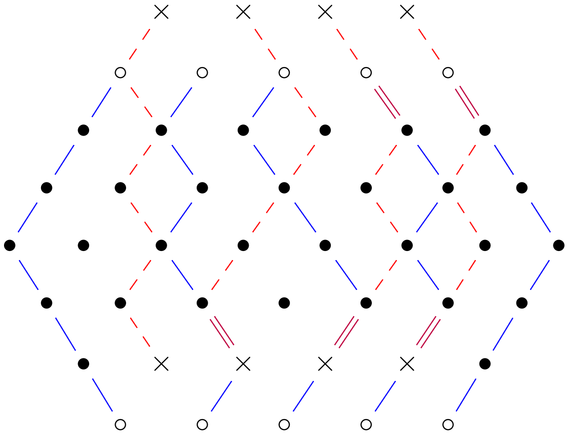
$$A \parallel B = \frac{1}{\frac{1}{A} + \frac{1}{B}}.$$

By induction on k, and the fact that we apply birational rowmotion from top to bottom, we can apply algebraic manipulations to reduce our result to proving the following **Plücker-like identity**:

$$\begin{aligned} \varphi_k(i-k,j-k)\varphi_{k-1}(i-k+1,j-k+1) = \\ \varphi_k(i-k,j-k+1)\varphi_{k-1}(i-k+1,j-k) \\ +\varphi_k(i-k+1,j-k)\varphi_{k-1}(i-k,j-k+1). \end{aligned}$$

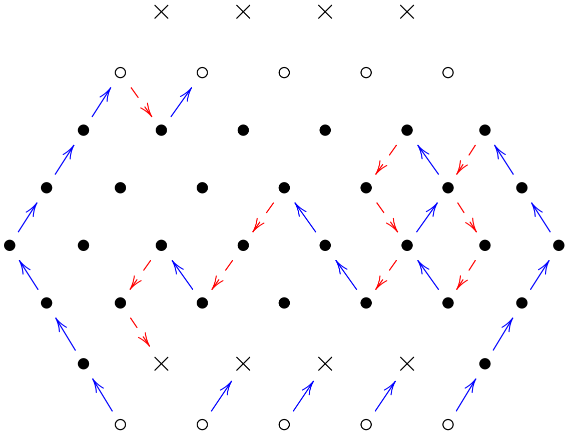
$$\varphi_k(i-k, j-k)\varphi_{k-1}(i-k+1, j-k+1) = \varphi_k(i-k, j-k+1)\varphi_{k-1}(i-k+1, j-k) + \varphi_k(i-k+1, j-k)\varphi_{k-1}(i-k, j-k+1).$$

Example (k=5):



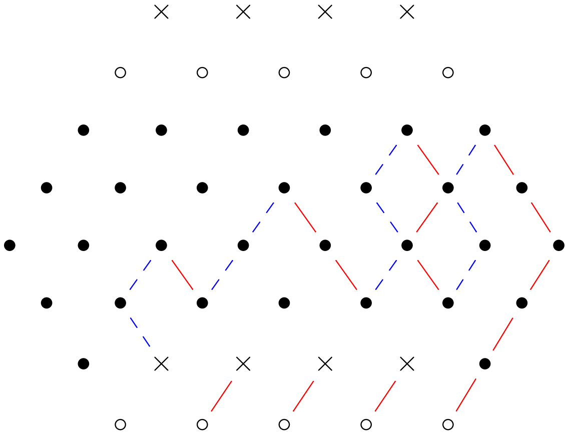
We build **bounce paths** and **twigs** (paths of length one from $\circ$ to $\times$) starting from the bottom row of $\circ$'s.

Example (k=5):



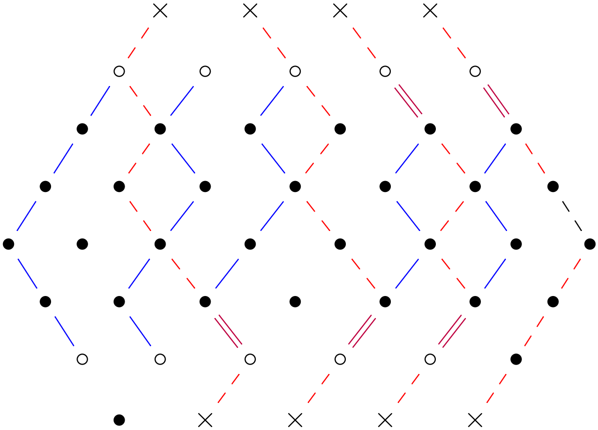
We then reverse the colors along the $(k-2)$ **twigs** and the **one bounce path from** $\circ$ **to** $\times$ (rather than $\circ$ to $\circ$).

Example (k=5):



Swap in the new colors and shift the ∘'s and ×'s in the bottom two rows.

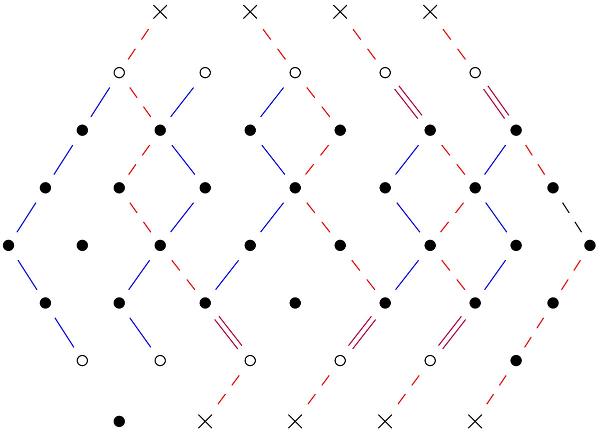
Example (k=5):



Sketch of Proof

$$\varphi_k(i-k,j-k)\varphi_{k-1}(i-k+1,j-k+1) = \varphi_k(i-k,j-k+1)\varphi_{k-1}(i-k+1,j-k) + \varphi_k(i-k+1,j-k)\varphi_{k-1}(i-k,j-k+1).$$

Example (k=5):



Further Application: Birational File Homomesy

Theorem

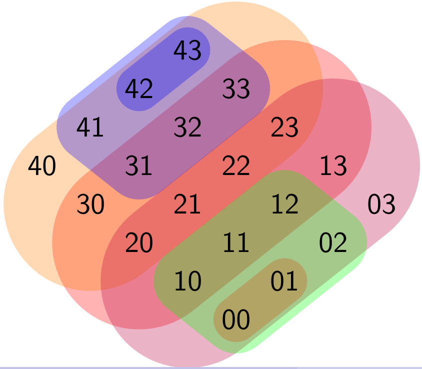
Given a file (i.e. a column) F in $[0,r]\times[0,s]$, $\prod_{k=0}^{r+s+1}\prod_{(i,j)\in F}\rho_B^k(i,j)=1.$

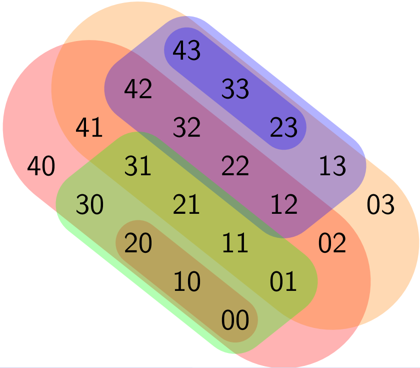
Further Application: Birational File Homomesy

Theorem

Given a file (i.e. a column) F in $[0,r] \times [0,s]$, $\prod_{k=0}^{r+s+1} \prod_{(i,j) \in F} \rho_B^k(i,j) = 1.$

Sketch of Proof: Double-counting argument, followed by color-coded cancellations and several entries immediately equal to 1, as in ensuing table.





Let $(r, s) = (4, 3)$, $d = 2$, and consider the file $F = \{(4, 2), (3, 1), (2, 0)\}$. The following table displays the values of $\rho_B^k(i, j)$ for $0 \le k \le 8$, $(i, j) \in F$.

	(4, 2)	(3, 1)	(2, 0)
$k = 0$	$\frac{\varphi_0(4,2)}{\varphi_1(4,2)=1}$	$\frac{\varphi_0(3,1)}{\varphi_1(3,1)}$	$\frac{\varphi_0(2,0)}{\varphi_1(2,0)}$
$k = 1$	$\frac{\varphi_1(3,1)}{\varphi_2(3,1)=1}$	$\frac{\varphi_1(2,0)}{\varphi_2(2,0)}$	$\mu^{(1,0)}\left[\frac{\varphi_0(2,0)}{\varphi_1(2,0)}\right]$
$k = 2$	$\frac{\varphi_2(2,0)}{\varphi_3(2,0)=1}$	$\mu^{(1,0)}\left[\frac{\varphi_1(2,0)}{\varphi_2(2,0)}\right]$	$\mu^{(2,0)}\left[\frac{\varphi_0(2,0)}{\varphi_1(2,0)}\right] = \frac{1}{x_{23}}$
$k = 3$	$\mu^{(1,0)}\left[\frac{\varphi_2(2,0)}{\varphi_3(2,0)=1}\right]$	$\mu^{(2,0)}\left[\frac{\varphi_1(2,0)}{\varphi_2(2,0)}\right]$	$\frac{\varphi_1(2,3)=1}{\varphi_0(2,3)}$
$k = 4$	$\mu^{(2,0)}\left[\frac{\varphi_2(2,0)}{\varphi_3(2,0)=1}\right]$	$\mu^{(3,1)}\left[\frac{\varphi_0(3,1)}{\varphi_1(3,1)}\right] = \frac{1}{x_{12}}$	$\frac{\varphi_2(1,2)=1}{\varphi_1(1,2)}$
$k = 5$	$\mu^{(3,1)}\left[\frac{\varphi_1(3,1)}{\varphi_2(3,1)=1}\right]$	$\frac{\varphi_1(1,2)}{\varphi_0(1,2)}$	$\frac{\varphi_3(0,1)=1}{\varphi_2(0,1)}$
$k = 6$	$\mu^{(4,2)}\left[\frac{\varphi_0(4,2)}{\varphi_1(4,2)=1}\right] = \frac{1}{x_{01}}$	$\frac{\varphi_2(0,1)}{\varphi_1(0,1)}$	$\mu^{(0,1)}\left[\frac{\varphi_3(0,1)=1}{\varphi_2(0,1)}\right]$
$k = 7$	$\frac{\varphi_1(0,1)}{\varphi_0(0,1)}$	$\mu^{(0,1)}\left[\frac{\varphi_2(0,1)}{\varphi_1(0,1)}\right]$	$\mu^{(1,2)}\left[\frac{\varphi_2(1,2)=1}{\varphi_1(1,2)}\right]$
$k = 8$	$\mu^{(0,1)}\left[\frac{\varphi_1(0,1)}{\varphi_0(0,1)}\right] = x_{42}$	$\mu^{(1,2)}\left[\frac{\varphi_1(1,2)}{\varphi_0(1,2)}\right] = x_{31}$	$\mu^{(2,3)}\left[\frac{\varphi_1(2,3)=1}{\varphi_0(2,3)}\right] = x_{20}$

Thanks for Listening!

Andries E. Brouwer and A. Schrijver, *On the period of an operator, defined on antichains*, Math. Centr. report ZW24, Amsterdam (Jun. 1974), http://www.win.tue.nl/~aeb/preprints/zw24.pdf.

Peter J. Cameron and Dmitry G. Fon-der-Flaass, *Orbits of Antichains Revisited*, Europ. J. of Combin., **16(6)**, (1995), 545–554, http://www.sciencedirect.com/science/article/pii/0195669895900365.

David Einstein and James Propp, *Combinatorial, piecewise-linear, and birational homomesy for products of two chains* (2013), arXiv:1310.5294.

David Einstein and James Propp, *Piecewise-linear and birational toggling (Extended abstract)*, DMTCS proc. FPSAC 2014, http://www.dmtcs.org/dmtcs-ojs/index.php/proceedings/article/view/dmAT0145/4518. Also available at arXiv:1404.3455v1.

Dmitry G. Fon-der-Flaass, *Orbits of Antichains in Ranked Posets*, Europ. J. Combin., **14(1)**, (1993), 17–22, http://www.sciencedirect.com/science/article/pii/S0195669883710036.

Darij Grinberg and Tom Roby, *The order of birational rowmotion (Extended abstract)*, DMTCS proc. FPSAC 2014, http://www.dmtcs.org/pdfpapers/dmAT0165.pdf.

Darij Grinberg and Tom Roby, *Iterative properties of birational rowmotion I: generalities and skeletal posets*, Electron. J. of Combin. **23**(1), #P1.33 (2016). http://www.combinatorics.org/ojs/index.php/eljc/article/view/v23i1p33

Thanks for Listening!

- Darij Grinberg and Tom Roby, *Iterative properties of birational rowmotion II: rectangles and triangles*, Electron. J. of Combin. **22**(3), #P3.40 (2015). `http://www.combinatorics.org/ojs/index.php/eljc/article/view/v22i3p40`.
- Gregg Musiker and Tom Roby, *Paths to understanding binational rowmotion on products of two chains* (2018). `arXiv:1801.03877v1`.
- James Propp, Tom Roby, Jessica Striker, and Nathan Williams (organizers), Sam Hopkins (notetaker), *Notes from the AIM workshop on dynamical algebraic combinatorics*, American Institute of Math., San Jose, CA, 23–27 March 2015, `http://aimath.org/pastworkshops/dynalgcomb.html`, `http://mit.edu/~shopkins/docs/aim_dyn_alg_comb_notes.pdf`.
- James Propp and Tom Roby, *Homomesy in products of two chains*, Electron. J. Combin. **22(3)** (2015), #P3.4,
- Tom Roby, *Dynamical algebraic combinatorics and the homomesy phenomenon* in Andrew Beveridge, et. al., Recent Trends in Combinatorics, IMA Volumes in Math. and its Appl., **159** (2016), 619–652.
- William A. Stein et. al., *Sage Mathematics Software (Version 8.1)*, The Sage Development Team (2017), `http://www.sagemath.org`.

Thanks for Listening!

Jessica Striker and Nathan Williams, *Promotion and Rowmotion*, Europ. J. of Combin. 33 (2012), 1919–1942, http://www.sciencedirect.com/science/article/pii/S0195669812000972. Also available at arXiv:1108.1172v3.

HughThomas and Nathan Williams, *Rowmotion in Slow Motion*, arXiv:1712.10123.

Emine Yıldırım, *Coxeter transformation on Cominuscule Posets*, arXiv:1710.10632.

Thanks for your attention!