Increasing Labelings, Generalized Promotion, and Rowmotion

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Outline

Outline

1. Background
2. Previous Results
3. Generalized Increasing Labelings

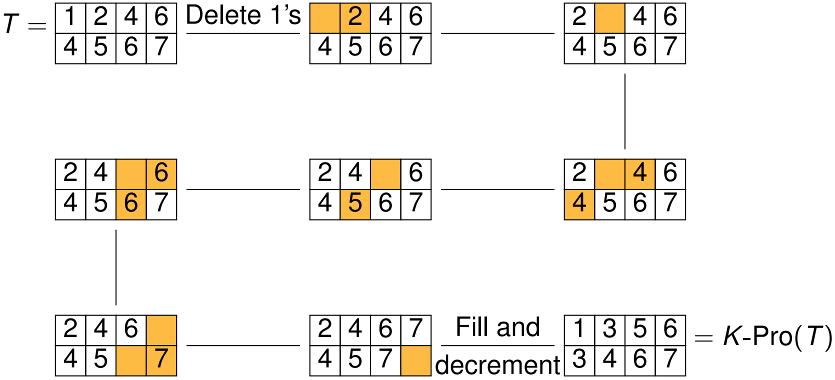
Increasing Tableaux

1	4	5	8
2	5	7	9
6	7	9	10
8	10		

Figure: An increasing tableau T of shape $\lambda = (4, 4, 4, 2)$.

Arise in equivariant K-theory of the Grassmanian.

K-Promotion



Order of K-promotion

- Order of K-promotion on rectangles $a \times b$ is always small multiple of largest possible label q.
- Hits it on the nose for special cases ($a = 2$, $q = a + b$, standard tableaux).

Rowmotion

Definition

Let *rowmotion* be the action on $J(P)$ (set of order ideals of P) that takes in an order ideal I, and returns the order ideal generated by the minimal elements of $J(P)$.

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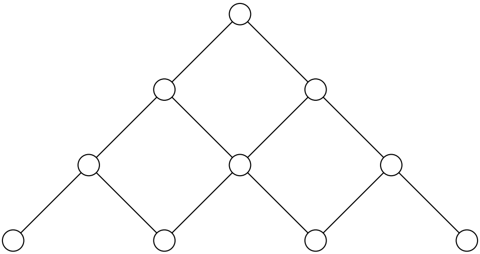

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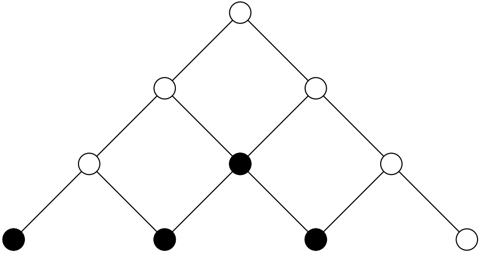

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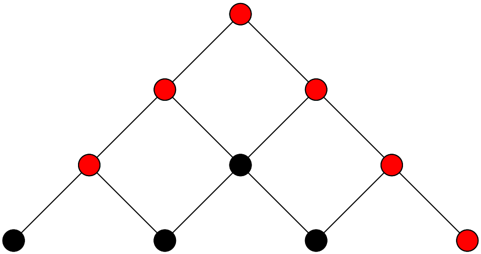

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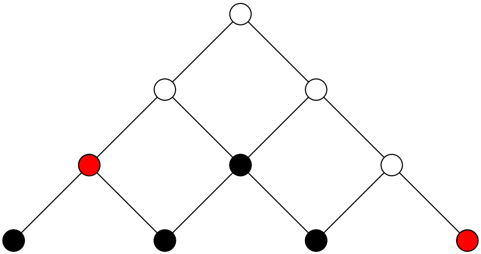

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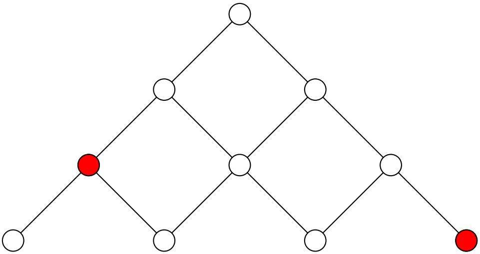

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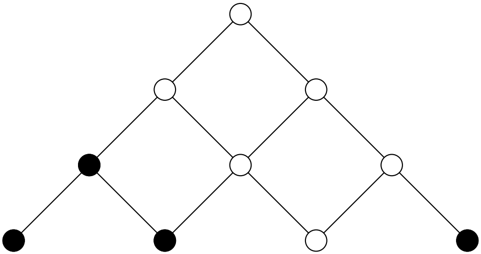


Toggle Group

Definition

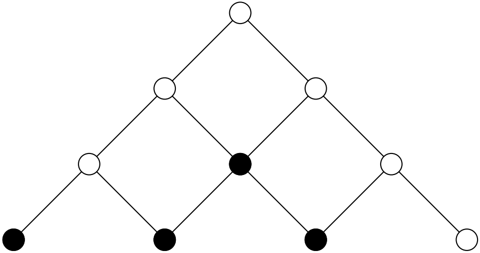
For each element $e \in P$ define its **toggle** $t_e : J(P) \to J(P)$ as follows.

$$t_e(X) = \begin{cases} X \cup \{e\} & \text{if } e \notin X \text{ and } X \cup \{e\} \in J(P) \\ X \setminus \{e\} & \text{if } e \in X \text{ and } X \setminus \{e\} \in J(P) \\ X & \text{otherwise} \end{cases}$$

Note: t_e, t_f commute whenever neither e nor f covers the other.

Theorem (Cameron–Fon-der-Flaass)

Rowmotion is the same as toggling each element once, in the order of a linear extension.



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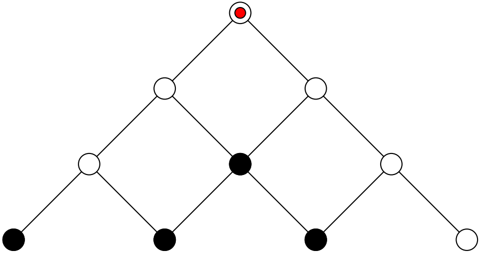

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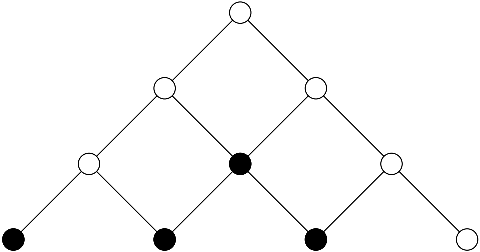

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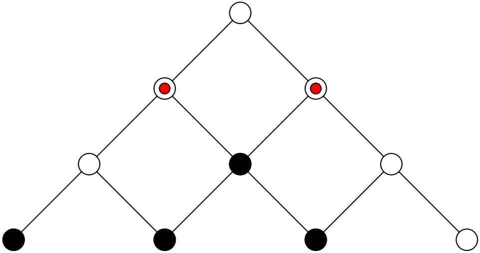

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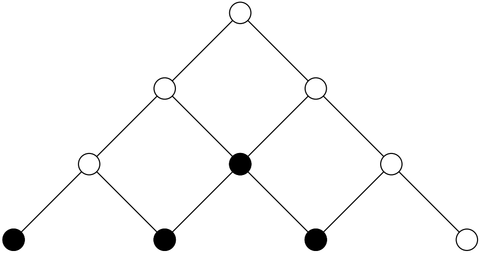

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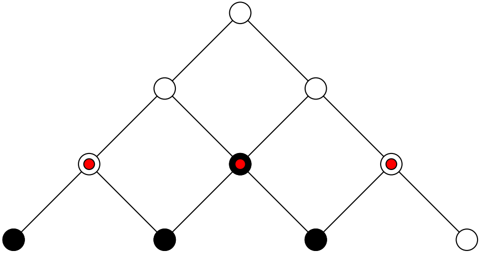

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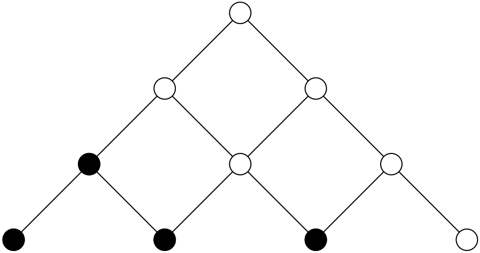

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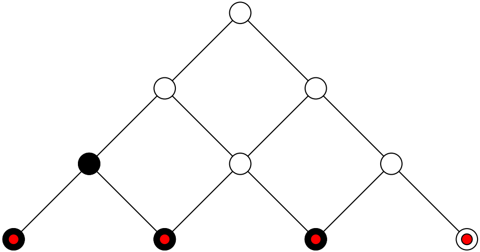

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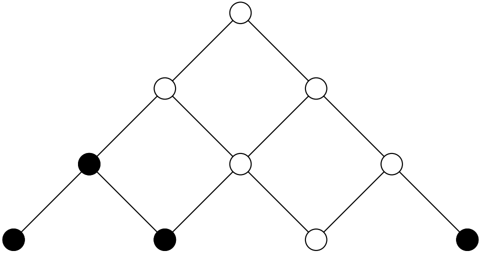

Order of Rowmotion

- Rowmotion on order ideals in $a \times b \times c$.
- When $c = 1, 2(, 3?)$, order is exactly $a + b + c - 1$.
- In general, orbit sizes appear to be small multiples of $a + b + c - 1$.

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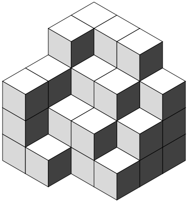
Bijection Between Increasing Tableaux and Order Ideals

Theorem (Dilks, Pechenik, Striker)

Bijection between order ideals in $[a] \times [b] \times [c]$ *and increasing labelings of* $a \times b$ *with max entry* $a + b + c - 1$*.*

Bijection Between Increasing Tableaux and Order Ideals

$I =$



Project to bottom face

4	4	4	3
4	3	3	2
3	2	2	1
3	1	0	0

Rotate 180°

0	0	1	3
1	2	2	3
2	3	3	4
3	4	4	4

Add 1+rank

1	2	4	7
3	5	6	8
5	7	8	10
7	9	10	11

$= \Psi_3(I)$

K-Bender-Knuth Involutions

Theorem (Dilks, Pechenik, Striker)

K-promotion is a series of local involutions.

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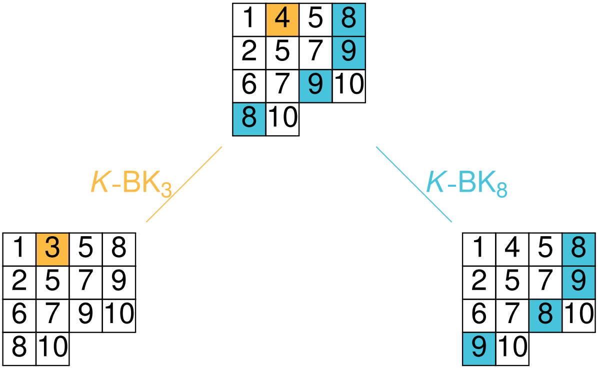
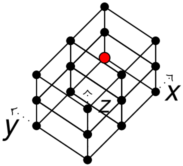
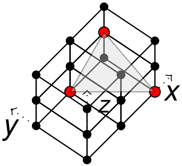


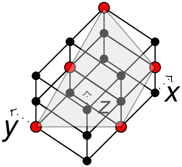
Equivariant Bijection



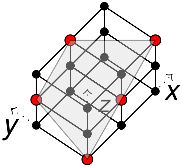
$x + y - z = 3$



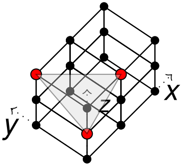
$x + y - z = 2$



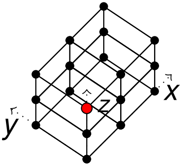
$x + y - z = 1$



$x + y - z = 0$



$x + y - z = -1$



$x + y - z = -2$

Hyperplane Promotion is Conjugate to Rowmotion

Theorem (Dilks, Pechenik, Striker)

Let P be a poset with an order and rank preserving map $\pi : P \to \mathbb{Z}^n$, and let $v = (v_1, v_2, v_3, \ldots, v_n)$, where $v_j \in \{\pm 1\}$.

Let $T^i_{\pi,v}$ be the product of toggles t_x for all elements x of P that lie on the affine hyperplane $\langle \pi(x), v \rangle = i$.

Then $\mathrm{Pro}_{\pi,v} = \ldots T^{-2}_{\pi,v} T^{-1}_{\pi,v} T^{0}_{\pi,v} T^{1}_{\pi,v} T^{2}_{\pi,v} \ldots$ is conjugate to rowmotion.

Punchline

K-promotion on increasing labelings of $a \times b$ with max entry $a + b + c - 1$

has the same orbit structure as

rowmotion on order ideals in $a \times b \times c$.

Outline

Motivation

How much do these methods generalize.

- Non-square increasing tableaux?
- Non-ranked posets?

Increasing Labeling

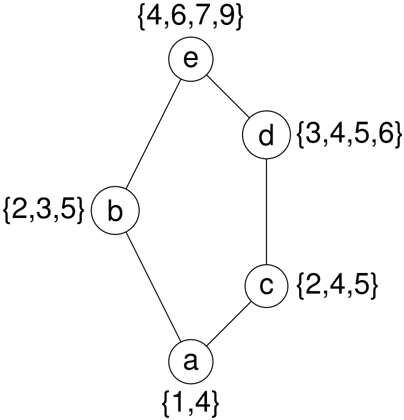
Definition

An *increasing labeling* of P is a function $f : P \to \mathbb{Z}$ such that $p_1 <_P p_2$ implies $f(p_1) < f(p_2)$.

Definition

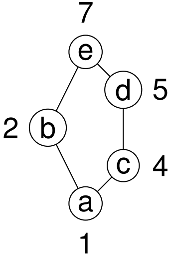
For labeling function $R : P \mapsto \mathcal{P}(\mathbb{Z})$, let $\mathrm{Inc}^R(P)$ be the set of increasing labelings of P such that for all $p \in P$, $f(p) \in R(p)$.

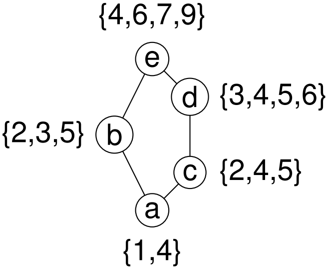
Example

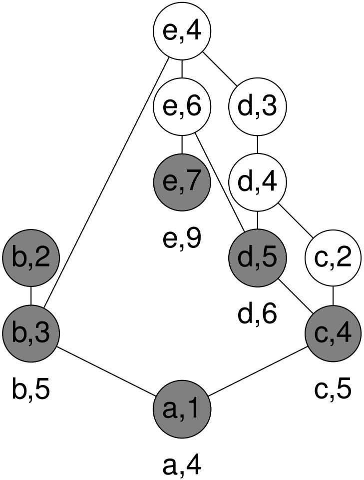


Key Idea

- $\mathrm{Inc}^R(P)$ can be partially ordered by element-wise comparison.
- $\mathrm{Inc}^R(P)$ is a distributive lattice (meet and join are taking element-wise min/max)
- Birkhoff FTFDL (Fundamental Theorem of Finite Distributive Lattices)
- Find join irreducibles and their relative order ($\Gamma(P, R)$).

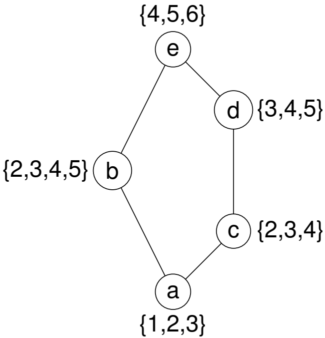

7
e
d
5
2
b
c
4
a
1

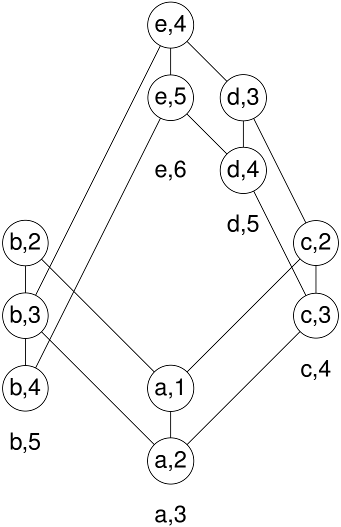

{4,6,7,9}
e
d
{3,4,5,6}
{2,3,5}
b
c
{2,4,5}
a
{1,4}


e,4
e,6
d,3
e,7
d,4
e,9
b,2
d,5
c,2
d,6
b,3
c,4
b,5
c,5
a,1
a,4

Special Case: Bounded max entry

Largest entry of 6





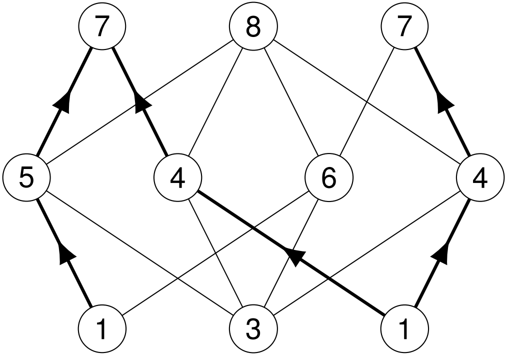
Generalized Promotion on Increasing Labelings

Generalized Bender-Knuth involutions for arbitrary R:

- If a label is currently i, and you can increment it to next allowable label (and stay in $\mathrm{Inc}^R(P)$), then do it.
- If you can decrement a label to become i (and stay in $\mathrm{Inc}^R(P)$), then do it.
- Otherwise, do nothing.

Increasing tableaux promotion:
$\text{IncPro} = \ldots \circ \rho_2 \circ \rho_1$.

- Generalizes promotion on linear extensions.
- Generalizes K-promotion on increasing tableaux.
- In case with largest global entry, can equivalently be described in terms of box sliding.



Toggle Promotion

Definition

Toggle order a function $H : P \to \mathbb{Z}$ where $p_1 \lessdot p_2 \implies H(p_1) \neq H(p_2)$.

Definition

T_H^i is the product of all t_p for $p \in P$ such that $H(p) = i$.

Toggle-promotion(wrt H), called TogPro_H, is the toggle group action given by

$$\dots T_H^2 T_H^1 T_H^0 T_H^{-1} T_H^{-2} \dots$$

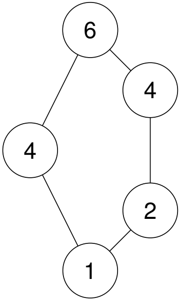
Bijection is equivariant

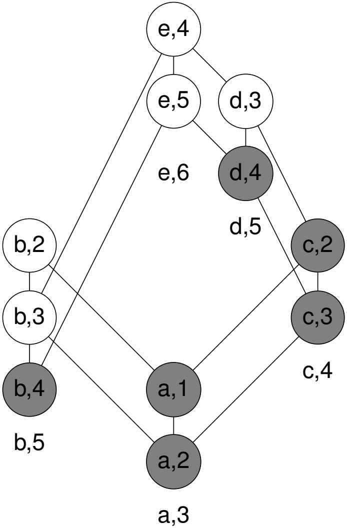
For $\Gamma(P, R)$, a natural toggle order is given by $H((p, k)) = k$.

Theorem (Dilks, Striker, Vorland)

The map between $\mathrm{Inc}^R(P)$ *and* $J(\Gamma(P, R))$ *equivariantly takes* IncPro *to* TogPro_H*.*

ρ_i is T_H^i





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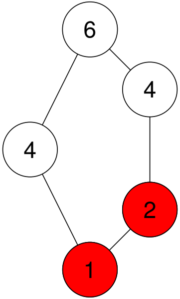
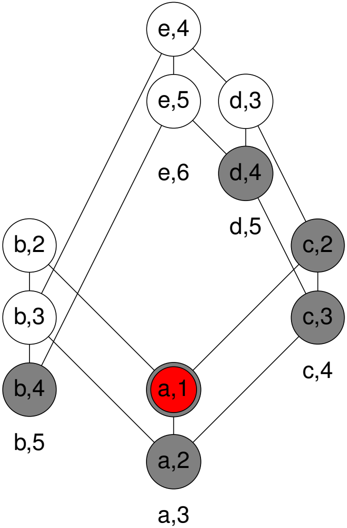
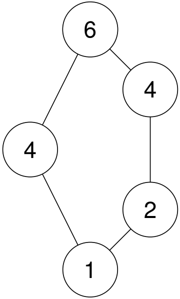
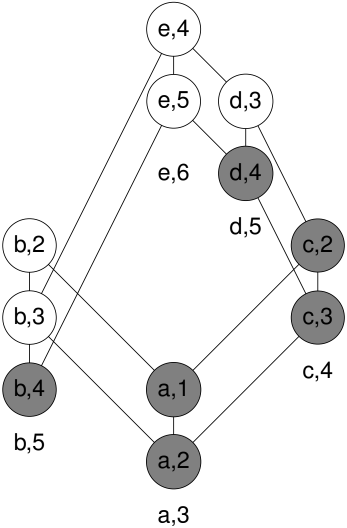




ρ_i is T^i_H





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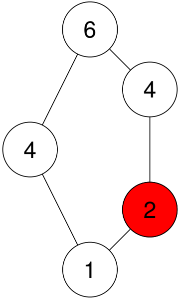

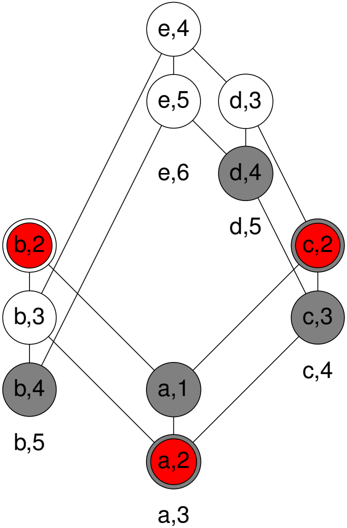

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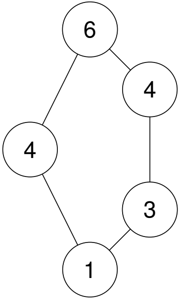

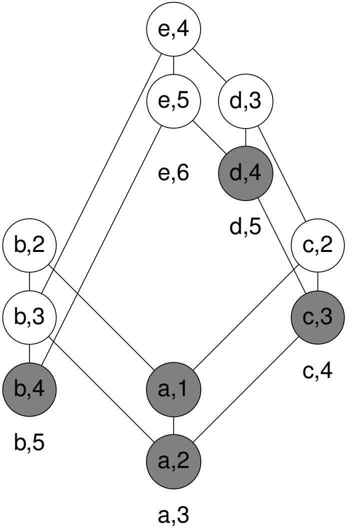

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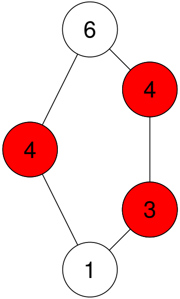
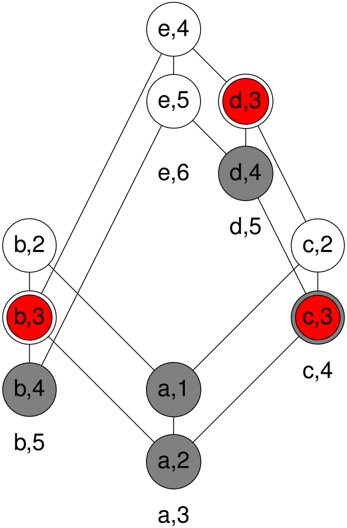
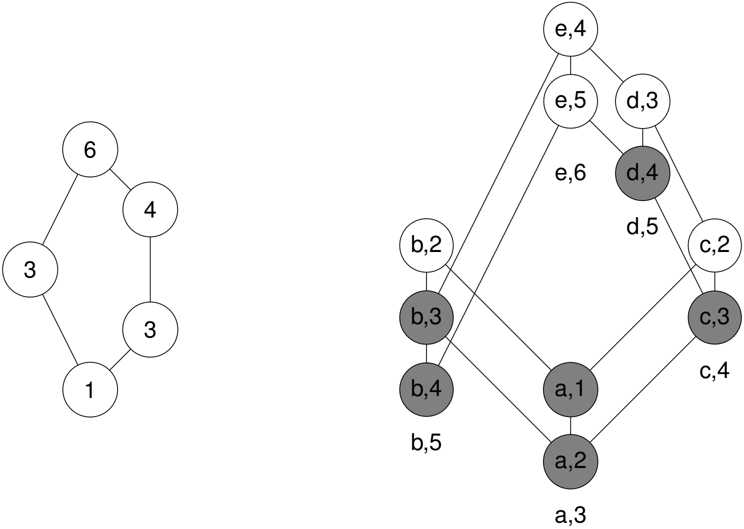




ρ_i is T_H^i



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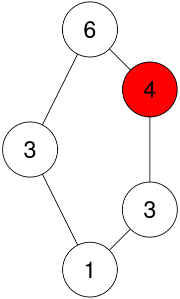

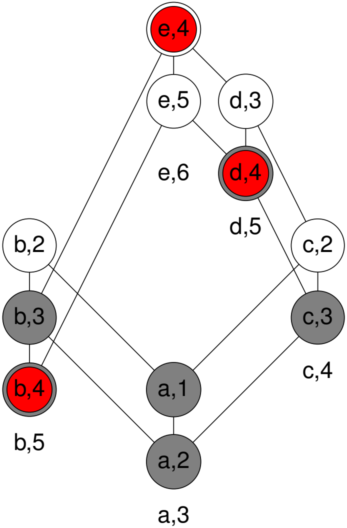

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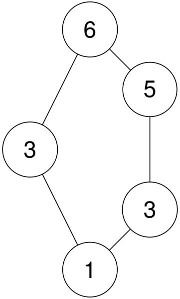

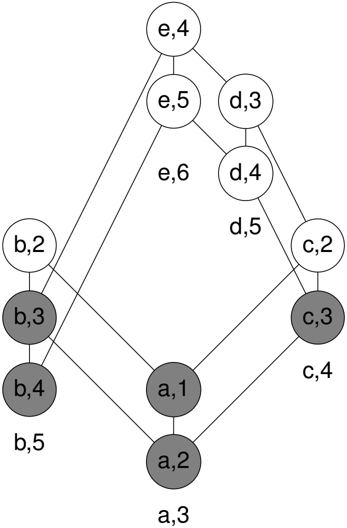

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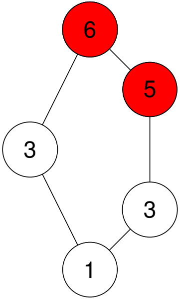
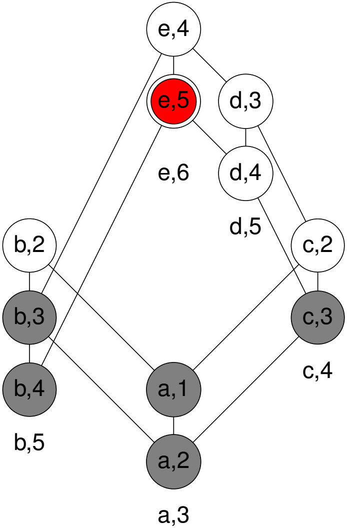
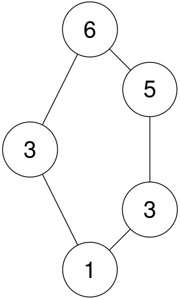
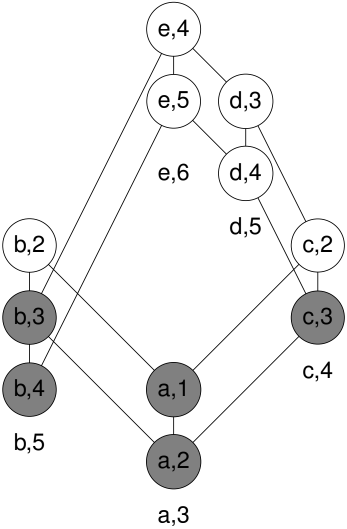




ρ_i is T^i_H





Column Toggle Order

Definition

We say that a function $H : P \to \mathbb{Z}$ is a *column toggle order* if whenever $p_1 \lessdot p_2$ in P, then $H(p_1) = H(p_2) \pm 1$.

Theorem (Dilks, Striker, Vorland)

When H is a column toggle order, then TogPro_H is conjugate to rowmotion.

Toggle Promotion (sometimes) conjugate to rowmotion

Theorem (Dilks, Striker, Vorland)

If R is a restriction function for P that consists of intervals (including global max entry), then the map $H : \Gamma(P,R) \to \mathbb{Z}$ given by $(p,k) \mapsto k$ is a column toggle order.

Therefore, rowmotion on $\Gamma(P,R)$ is conjugate to the corresponding toggle promotion.

Theorem

If P_1 and P_2 are ranked posets, then $H : P_1 \times P_2 :\to \mathbb{Z}$ given by $H((p_1,p_2)) = \mathrm{rk}_{P_1}(p_1) - \mathrm{rk}_{P_2}(p_2)$ is a column toggle order.

Big result

Theorem (Dilks, Striker, Vorland)

Increasing promotion on $\mathrm{Inc}^R(P)$ *is conjugate to rowmotion on* $\Gamma(P, R)$ *when* R *consists of intervals.*

Thanks!