The Coxeter-biCatalan Kreweras Complement

Emily Barnard

Northeastern University

Special Session on Dynamical Algebraic Combinatorics

A Lattice-Theoretic Kreweras Complement

Emily Barnard

Northeastern University

Special Session on Dynamical Algebraic Combinatorics

Lattice-theoretic background

Definition

A lattice L is a poset such that for each pair of elements u and w

- the smallest upper bound or **join** $u \vee w$ exists and
- the greatest lower bound or **meet** $u \wedge w$ exists.

Convention

We consider only finite lattices. We write $\hat{1}$ for the top element and $\hat{0}$ for the bottom element.

Definition

An element $j \in L$ is **join-irreducible** if $j = \bigvee A$ implies $j \in A$.

The canonical join representation (CJR)

The **canonical join representation** of an element w in L is the unique lowest irredundant expression $\bigvee A = w$. More precisely:

- The expression $\bigvee A = w$ is a **join-representation** for w.
- The join $\bigvee A$ is **irredundant** if

$$\bigvee A' < \bigvee A \text{ for each proper subset } A' \subset A.$$

- Observe that if $\bigvee A$ is irredundant then A is an antichain.
- For $\bigvee A$ and $\bigvee B$ irredundant, we say A is "lower" than B if the order ideal generated by A is contained in the order ideal generated by B.

Examples

What is the canonical join representation for the top element?

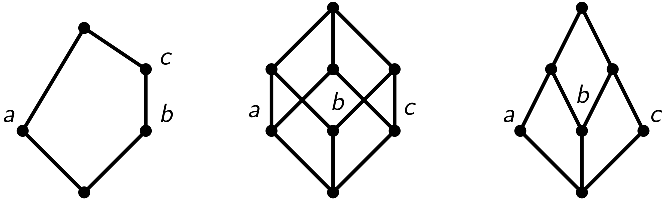


Figure: A Tamari lattice, a Boolean lattice, and the lattice L_6.

Observation

Each irredundant join of atoms is a canonical join representation.

Labeling the edges of L

Proposition [Barnard]

Suppose that $\bigvee A$ is the CJR of w. Then, for each $y \lessdot w$ there is a corresponding element $j \in A$ such that $j \vee y = w$.
Moreover, j is the unique minimal element in L with this property.
The map $y \mapsto j$ is a bijection.

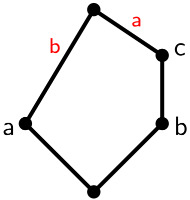


Figure: Labeling the edges in our Tamari lattice

Labeling the edges of L

Proposition [Barnard]

Suppose that $\bigvee A$ is the CJR of w. Then, for each $y \lessdot w$ there is a corresponding element $j \in A$ such that $j \vee y = w$.
Moreover, j is the unique minimal element in L with this property.
The map $y \mapsto j$ is a bijection.

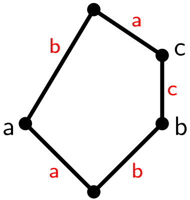


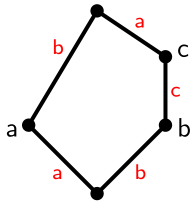
Figure: Labeling the edges in our Tamari lattice

The κ operation on L

Definition/Theorem [Barnard]

Define a map $\kappa : L \to L$ as follows:

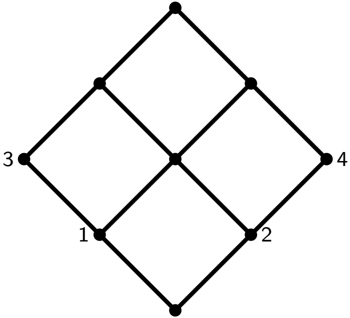
- Given $w \in L$, let A be the set of its down-edge labels.
- Let $\kappa(w)$ be the element whose up-edges are labeled by precisely the same set A.



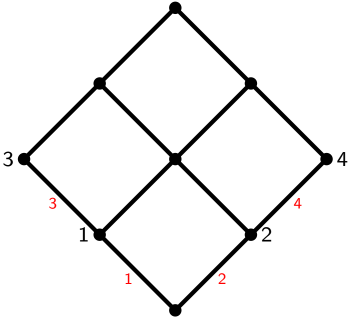
$$\kappa(\hat{1}) = \hat{0}$$

$$\kappa(a) = c \quad \kappa(c) = b \quad \kappa(b) = a$$

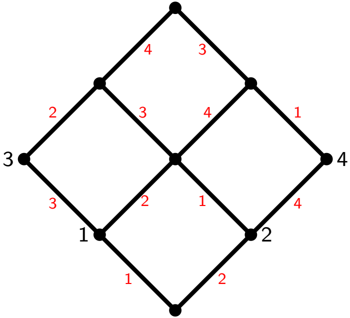
The κ operation "in nature"



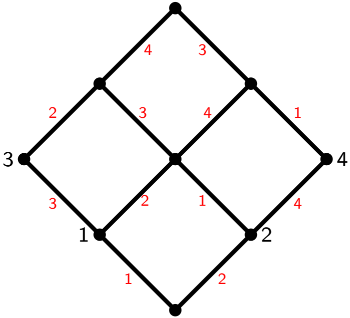
The κ operation "in nature"

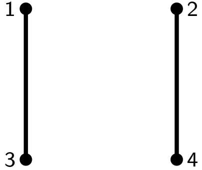


The κ operation "in nature"



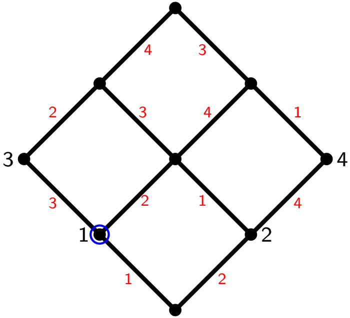
The κ operation "in nature"

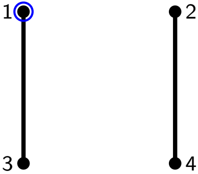




The dual of the poset $\mathcal{P}$ of join-irreducibles

The κ operation "in nature"

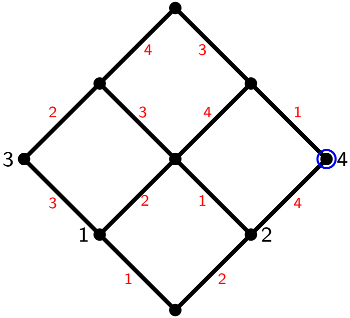


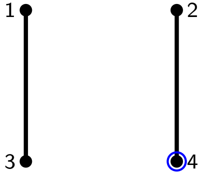


The dual of the poset $\mathcal{P}$ of join-irreducibles

Apply κ to 1

The κ operation "in nature"

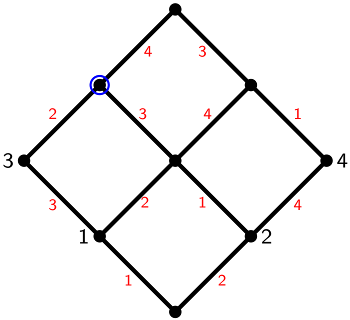


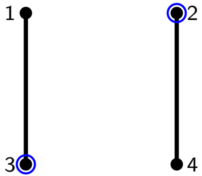


The dual of the poset $\mathcal{P}$ of join-irreducibles

$\kappa(1) = 4$

The κ operation "in nature"



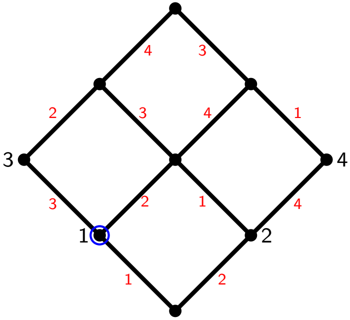


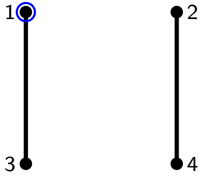
The dual of the poset $\mathcal{P}$ of join-irreducibles

$$\kappa(1) = 4$$

$$\kappa(4) = \bigvee\{2, 3\}$$

The κ operation "in nature"





The dual of the poset $\mathcal{P}$ of join-irreducibles

$$\kappa(1) = 4$$

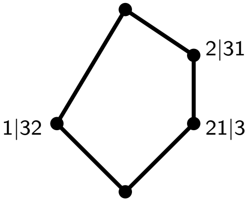
$$\kappa(4) = \bigvee\{2,3\}$$

$$\kappa(\bigvee\{2,3\}) = 1$$

Applying κ = Rowmotion on the dual of $\mathcal{P}$

The κ operation "in nature"

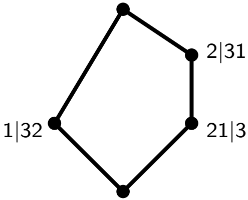
- Reading constructed an explicit bijection from c-sortable elements to noncrossing partitions.
- This bijection is essentially $w \mapsto \mathrm{CJR}(w)$.

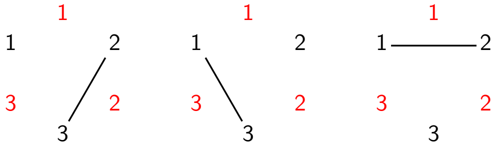


$\kappa(1|32) = 2|31$ $\kappa(2|31) = 21|3$

$\kappa(21|3) = 1|32$

The κ operation "in nature"





$\kappa(1|32) = 2|31$ $\kappa(2|31) = 21|3$

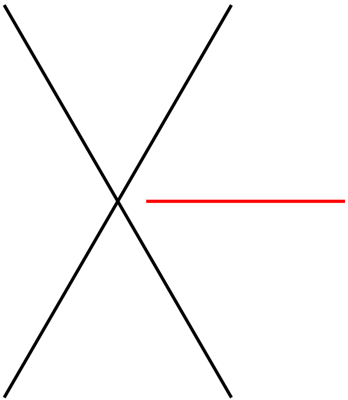
$\kappa(21|3) = 1|32$

<u>κ = Kreweras Complement</u>

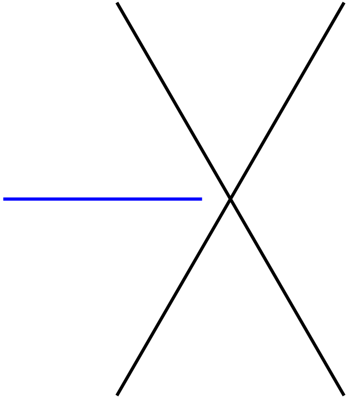
Coxeter biCatalan Combinatorics

W-Catalan	**W-biCatalan**
Vertices of the generalized associahedron	Vertices of the generalized **bi**-associahedron
c-sortable elements in (W, S)	c-**bi**sortable elements in (W, S)
Elements in $NC(W, c)$	Certain **pairs** (x, y): $x \in NC(W, c)$ and $y \in NC(W, c^{-1})$
Elements in **Camb**(W, c)	Elements in the c-biCambrian lattice, denoted **biCamb**(W, c)
Antichains in the root poset	Antichains in the **doubled** root poset

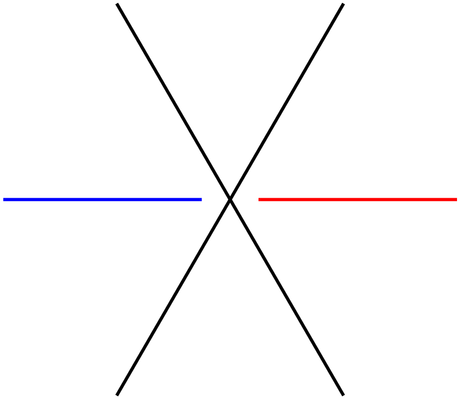
A biCambrian fan/lattice



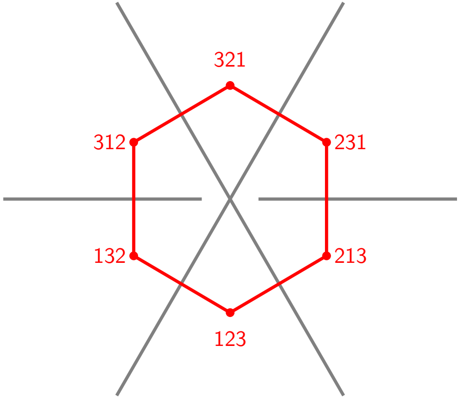
A biCambrian fan/lattice



A biCambrian fan/lattice



A biCambrian fan/lattice



The type A_3 biCambrian fan/lattice

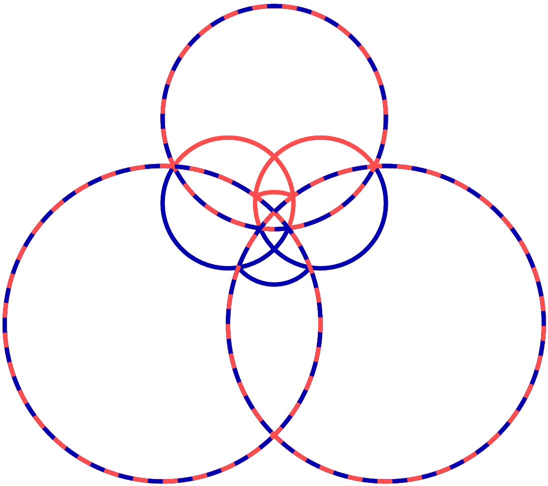


Figure: The bipartite biCambrian fan in type A_3

When you have a hammer...

Definition

- Let $f : \textbf{biCamb}(W, c) \to \mathbb{R}$ be the statistic

$$f(w) = \text{the cardinality of the CJR}(w)$$

Theorem

Let W be a finite irreducible Coxeter group of rank n and let $\textbf{biCamb}(W, c)$ be the bipartite biCambrian lattice of type W. Then the triple $(\textbf{biCamb}(W, c), f, \kappa)$ is $n/2$-homomesic.

When you have a hammer...

- Define $\mathrm{biNC}(W, c)$ to be the subposet of the shard intersection order induced to the set of c-bisortable elements.
- Let $\kappa : \mathrm{biNC}(W, c) \to \mathrm{biNC}(W, c)$ be induced from the lattice-theoretic κ operation acting on $\mathbf{biCamb}(W, c)$.

Theorem

Let c be a bipartite Coxeter element, and let W be a finite irreducible Coxeter group of rank n. Then $\mathrm{biNC}(W, c)$ satisfies:

1. $\mathrm{biNC}(W, c)$ is self-dual.
2. $\mathrm{biNC}(W, c)$ is ranked.
3. $\mathrm{rk}(w) = n - \mathrm{rk}(\kappa(w))$
4. κ is a **lattice complement** on $\mathrm{biNC}(W, c)$ meaning that

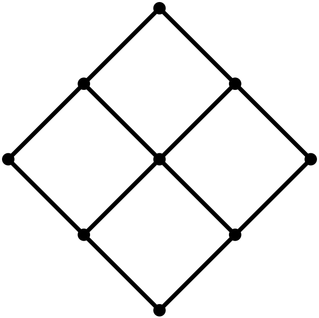
$$\kappa(w) \wedge w = \hat{0} \text{ and } \kappa(w) \vee w = \hat{1}.$$

Open Questions: The Doubled Root Poset

W-Catalan	**W-biCatalan**
Vertices of the generalized associahedron	Vertices of the generalized **bi**-associahedron
c-sortable elements in (W, S)	c-**bi**sortable elements in (W, S)
Elements in $NC(W, c)$	Certain **pairs** (x, y): $x \in NC(W, c)$ and $y \in NC(W, c^{-1})$
Elements in **Camb**(W, c)	Elements in the c-biCambrian lattice, denoted **biCamb**(W, c)
Antichains in the root poset	Antichains in the **doubled** root poset

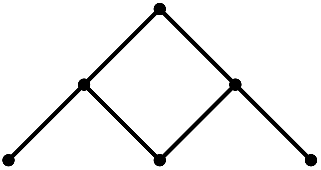
Open Question: The Doubled Root Poset

The doubled root poset in type A_3:



Open Question: The Doubled Root Poset

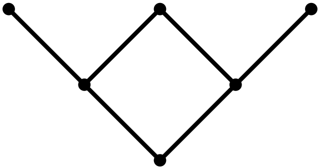
The doubled root poset in type A_3:



The root poset of type A_3

Open Question: The Doubled Root Poset

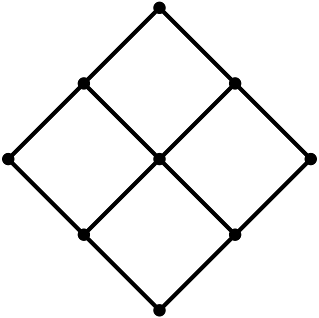
The doubled root poset in type A_3:



The dual of the root poset of type A_3

Open Question: Doubled Root Poset

The doubled root poset in type A_3:



Glue together at the simples

Open Question: Doubled Root Poset

Theorem [Armstrong, Stump, Thomas]

The Kreweras complement acting on $NC(W, c)$ has the same orbit structure as romotion acting the antichains in the root poset.

Question

How do rowmotion-orbits of the antichains in the doubled root poset compare with κ-orbits of c-bisortable elements?

Thank you!