Permutations Realized by Dynamical Systems

Kate Moore

January 13, 2018

Overview

Example: $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 4x(1 - x)$, initial condition $x_0 = .10$.

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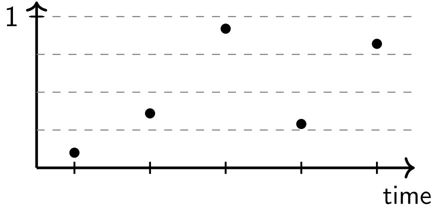
$f^3(x_0) = .29$

$f^4(x_0) = .82$



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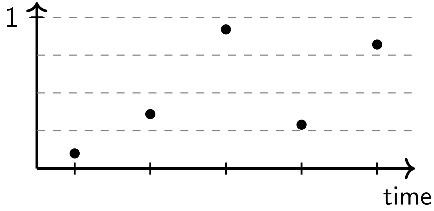
$x_0 = .10$	**1**
$f(x_0) = .36$	**3**
$f(f(x_0)) = .92$	**5**
$f^3(x_0) = .29$	**2**
$f^4(x_0) = .82$	**4**



The pattern of length 5 for f at $x_0 = .10$ is $\pi =$ **13524**.

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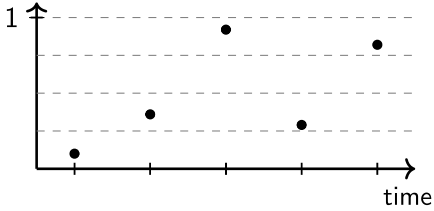
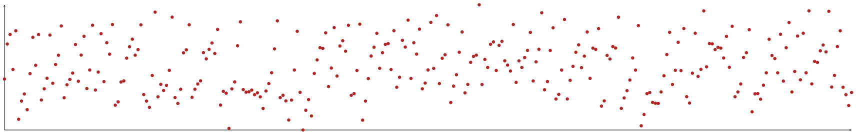
$$\left.\begin{aligned} x_0 &= .10 \\ f(x_0) &= .36 \\ f(f(x_0)) &= .92 \\ f^3(x_0) &= .29 \\ f^4(x_0) &= .82 \end{aligned}\right\} \begin{matrix} \mathbf{1} \\ \mathbf{3} \\ \mathbf{5} \\ \mathbf{2} \\ \mathbf{4} \end{matrix}$$



The pattern of length 5 for f at $x_0 = .10$ is $\pi = \mathbf{13524}$.

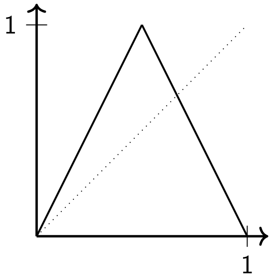


Motivation: Understand time series in the context of dynamical systems.

Allowed Patterns

What are the allowed patterns?

Example: Let $T(x) = \min\{2x, 2(1-x)\}$.



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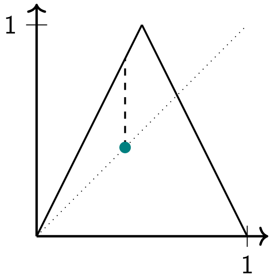
Example: Let $T(x) = \min\{2x, 2(1-x)\}$.

$(\mathbf{x}, \mathbf{T(x)}, \mathbf{T^2(x)}) = (.42, .84, .32) \rightsquigarrow \mathbf{231}$



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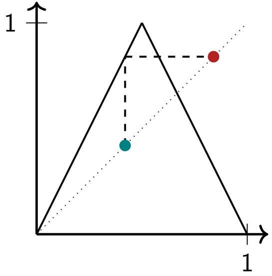

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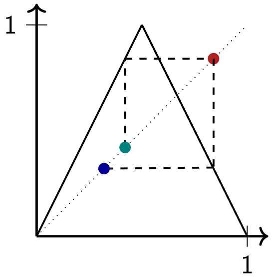

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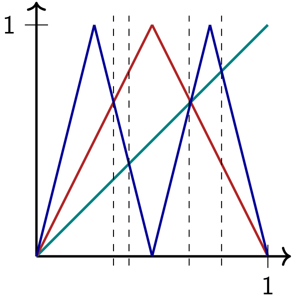


n	3	4	5	6	7	8	10	12
$\lvert \mathrm{Allow}_n(T) \rvert$	5	12	31	75	178	414	2137	10,525

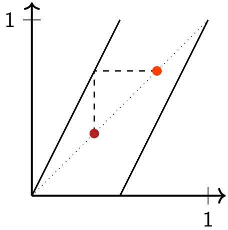
Best-known Bounds: (Elizalde & M.)

Notice: 321 is forbidden $\rightsquigarrow$ $\underline{321}4, \underline{421}3, 12\underline{653}4 \ldots$ are forbidden.

Permutation Structure of Periodic Points

Example: $F_2(x) = 2x \mod 1$. The point $x_0 = \frac{11}{31}$ is 5-periodic.

$$\left(\frac{11}{31}, \frac{22}{31}, \frac{13}{31}, \frac{26}{31}, \frac{21}{31}\right)^\infty$$



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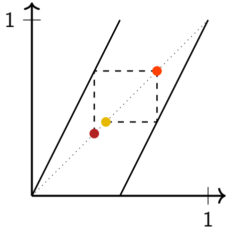
Other representatives of the periodic orbit give cyclic rotations of π.



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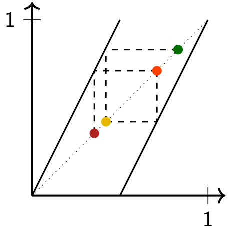

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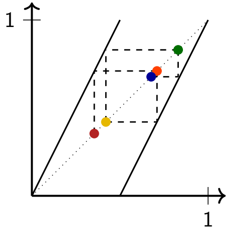

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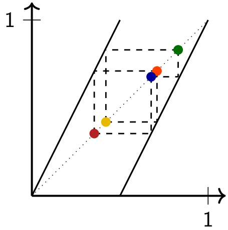

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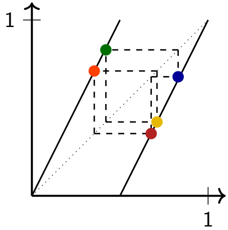

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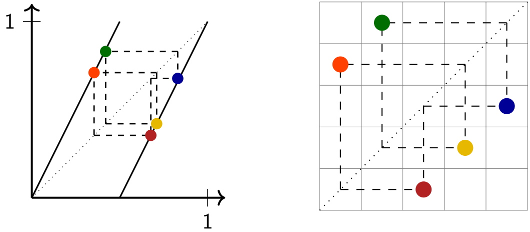


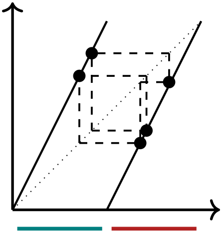
$$\hat{\pi} = (1, 4, 2, 5, 3) = 45123 \in \mathcal{C}_5$$

Key Idea: Cycles $\hat{\pi}$ obtained in this way have at most one descent.

Permutation Structure of Periodic Points

Let $F_2(x) = 2x \bmod 1$ and $\mathbf{I_0} = [\mathbf{0}, \frac{1}{2})$, $\mathbf{I_1} = [\frac{1}{2}, \mathbf{1})$.

The itinerary for the 5-periodic orbit of $\frac{11}{31}$ is:



$$\left(\frac{11}{31}, \frac{22}{31}, \frac{13}{31}, \frac{26}{31}, \frac{21}{31}\right)^{\infty} \to (\ \mathbf{0\ 1\ 0\ 1\ 1}\ )^{\infty}$$

Which is the binary expansion:

$$\frac{11}{31} = \frac{\mathbf{0}}{2} + \frac{\mathbf{1}}{2^2} + \frac{\mathbf{0}}{2^3} + \frac{\mathbf{1}}{2^4} + \frac{\mathbf{1}}{2^5} + \ldots.$$

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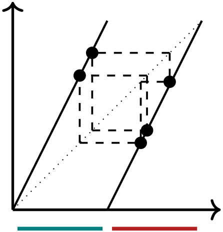

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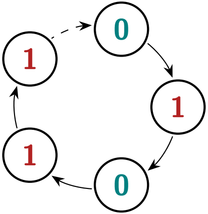
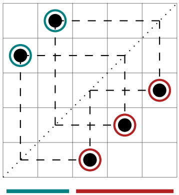
(Gessel & Reutenauer, '93)



$\Longleftrightarrow$

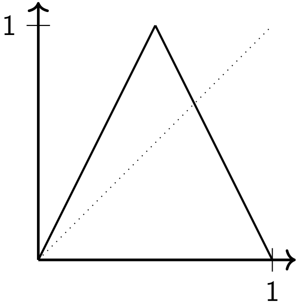


Binary necklaces with distinct rotations of length n

Cycles of length n with (at most) one descent

The Shape of Patterns

Example: $T(x) = \min\{2x, 2(1-x)\}$ and initial condition $x_0 = .42$.



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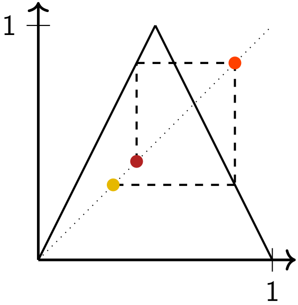

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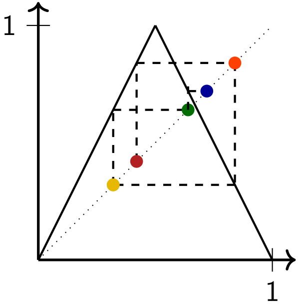

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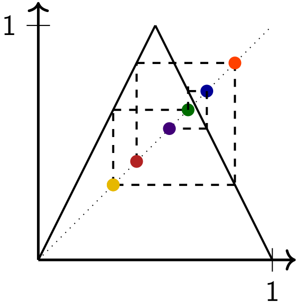

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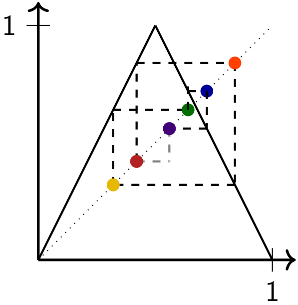

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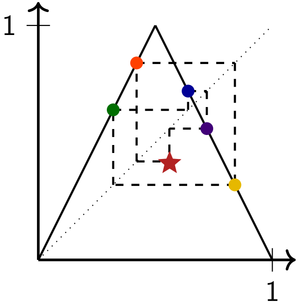

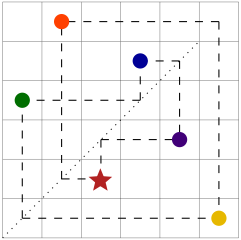

$$\pi = 261453 \longrightarrow \hat{\pi}^\star = (\star, 6, 1, 4, 5, 3) = 46\star531$$

Key Idea: If $\pi \in \text{Allow}_n(T)$, then $\hat{\pi}^\star$ is unimodal.

We ignore the value covered by $\star$.

(Except in the case that it is the first or last digit of $\hat{\pi}^\star$.)

Signed Shifts

The transformation:

$$\pi = \pi_1\pi_2\ldots\pi_n \longrightarrow \hat{\pi}^\star = (\star, \pi_2, \ldots, \pi_n) = \hat{\pi}_1^\star\hat{\pi}_2^\star\ldots\hat{\pi}_n^\star$$

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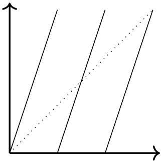
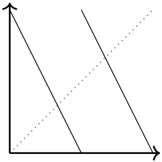
(Except in the case that it is the first or last digit of $\hat{\pi}^\star$.)

Theorem (Elizalde & M.): $\pi \in \mathrm{Allow}(\Sigma_\sigma)$ if and only if $\hat{\pi}^\star$ has the same monotonicity as σ and π is not "σ-collapsed."



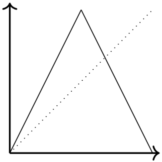
$\sigma = +++$

$F_3(x) = 3x \mod 1$



$\sigma = --$

$G_2(x) = -2x \mod 1$



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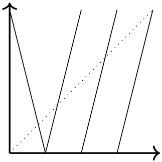
Tent Map



$\sigma = -+++$

Topological Entropy and Patterns

Suppose that $f : \mathbb{R} \to \mathbb{R}$ has finitely many monotone segments.

The *topological entropy* of f is

$$h^{\text{top}}(f) := \lim_{n \to \infty} \frac{1}{n} \log(c_n),$$

where c_n is the number of monotone segments of f^n.

Example: $T(x) = \min\{2x, 2(1-x)\}$

$$c_n = 2^n \quad \rightsquigarrow \quad h^{\text{top}}(T) = \log(2)$$

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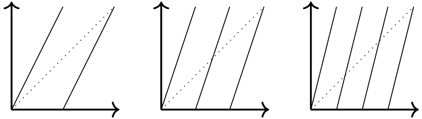
Theorem: (Bandt, Keller & Pompe '02)

$$h^{\text{top}}(f) = \lim_{n\to\infty} \frac{1}{n-1} \log(|\operatorname{Allow}_n(f)|).$$

Important Idea: We do not require knowledge of f to estimate the topological entropy using patterns.

Enumerating Allowed Patterns

$$b(n,k) := |\text{Allow}_n(F_k)| \setminus |\text{Allow}_n(F_{k-1})|$$



k	**2**	**3**	**4**	**5**
$b(6,k)$	126	402	186	6

Theorem (Elizalde):

$$\sum_{k=2}^{n} b(n,k)x^k = (1-x)^n \sum_{k\geq 2} p(n,k)x^k,$$

where

$$p(n,k) = \sum_{i=1}^{n-1} k^{n-i-1}\psi_k(i) + (k-2)k^{n-2},$$

and $\psi_k(i)$ is the number of k-ary primitive words of length i.

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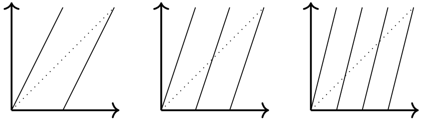

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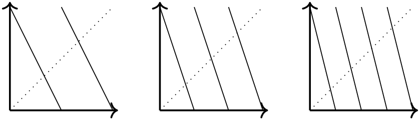
$\overline{b}(n,k) := |\mathrm{Allow}_n(G_k)| \smallsetminus |\mathrm{Allow}_n(G_{k-1})|$



k	**2**	**3**	**4**	**5**
$\overline{b}(6,k)$	140	408	168	4

Theorem (Elizalde & M.):

$$\sum_{k=2}^{n} \overline{b}(n,k)x^k = (1-x)^n \sum_{k\geq 2} \overline{p}(n,k)x^k.$$

where $\overline{p}(n,k)$ is equal to

$$\sum_{i=1}^{n-1} k^{n-i-1}\psi_k(i) + (k^2-2)k^{n-3} - 2\sum_{j=1}^{k-1} j^{n-3} - 2\sum_{\substack{c=1\\ \text{odd}}}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{j=1}^{k-1} \binom{c+k-j-2}{k-j} j^{n-2c-1}\psi_j(c),$$

and $\psi_k(i)$ is the number of k-ary primitive words of length i.

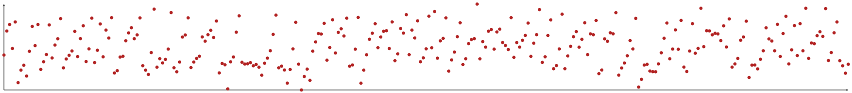
Enumerating Allowed Patterns (cont.)

$n \setminus k$	2	3	4	5	6	7
3	6					
4	18	6				
5	48	66	6			
6	**126**	**402**	**186**	**6**		
7	306	2028	2232	468	6	
8	738	8790	19426	10212	1098	6

$n \setminus k$	2	3	4	5	6	7
3	6					
4	20	4				
5	54	62	4			
6	**140**	**408**	**168**	**4**		
7	336	2084	2208	408	4	
8	800	9152	19580	9820	964	4

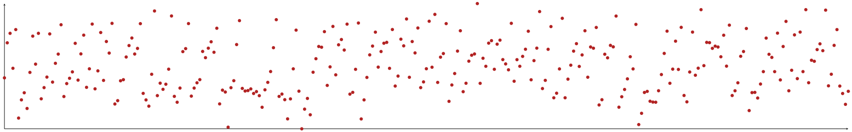
Values of $b(n,k)$ (left) and $\overline{b}(n,k)$ (right).

Corollary: The smallest forbidden patterns of $F_k(x) = kx \mod 1$ and $G_k(x) = -kx \mod 1$ are of length $k+2$.



Lower Bounds on Entropy Using Patterns

Proposed Goal: Use patterns to obtain lower bounds on entropy in certain settings (e.g. shifts, unimodal, continuous, ...).

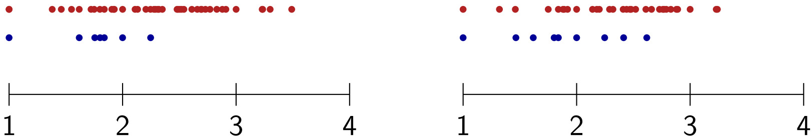


Relations with: Lower bound on entropy for any continuous map containing a periodic point with the given cycle structure, $\hat{\pi}$.

(Sarkovskii, Baldwin, and many others in the 80s).

Lower Bounds on Entropy Using Patterns

For $\beta > 1$, consider $F_\beta(x) = \beta x \mod 1$ and $G_\beta(x) = -\beta x \mod 1$.



The infimum β for which $\pi \in \mathrm{Allow}(F_\beta)$ (left) and $\pi \in \mathrm{Allow}(G_\beta)$ (right) for all π of length $n = 4, 5$.

(Elizalde), (Elizalde & M.; Charlier & Steiner)

Example: If we suppose our time series is generated by G_β and we observe $\pi = 15237864$, then we must have had

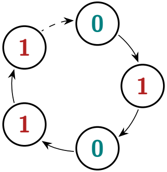
$$\beta \geq 3.154,$$

the largest real root of

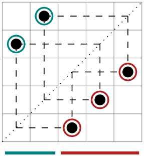
$$P_\pi(x) = x^4 - 4x^3 + 3x^2 - 2x + 3.$$

Thank You

Dynamical interpretation of combinatorial problems.



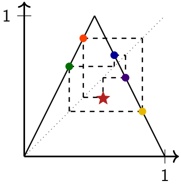
$\Longleftrightarrow$



Binary necklaces with distinct rotations of length *n*

Cycles of length *n* with (at most) one descent

Permutation-based techniques for estimating entropy of time series.



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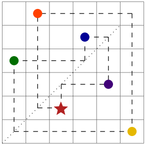

Permutation Entropy

The permutation entropy (re-scaled) of a time series $\{X_t\}_{t=1}^N$ is

$$\mathrm{PE}_n(X) := \frac{1}{n-1} \sum_{\pi \in \mathcal{S}_n} -p_\pi \log(p_\pi),$$

where p_π is the relative frequency of π in $\{X_t\}_{t=1}^N$.

Pattern analog of the metric entropy of iterated interval map [BKP].

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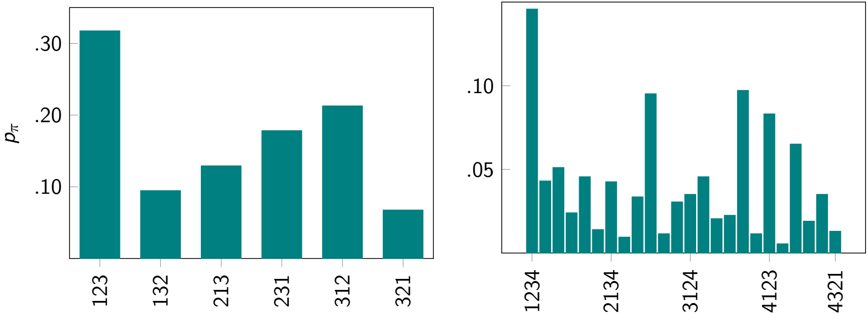

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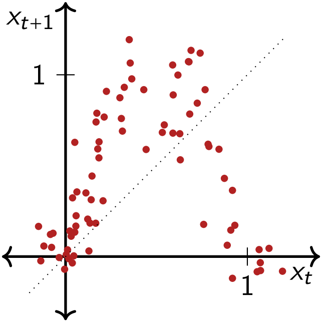

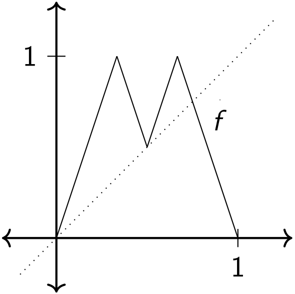

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In the case of an iterated map, $\mathrm{PE}_n(X)$ converges to the metric entropy of f:

$$h^{\mathrm{met}}(f) := \lim_{n \to \infty} \frac{1}{n} \sum_{I_K \in \mathcal{P}_n} -\mathbb{P}(I_K) \log(\mathbb{P}(I_K)),$$

where $\mathcal{P}_n$ is the set of montone segments of f^n and probabilities are determined by an invariant measure, $\mu(f^{-1}(A)) = \mu(A)$ [BKP].

Minimal Forbidden

The smallest forbidden patterns of $F_4(x) = 4x \bmod 1$ are

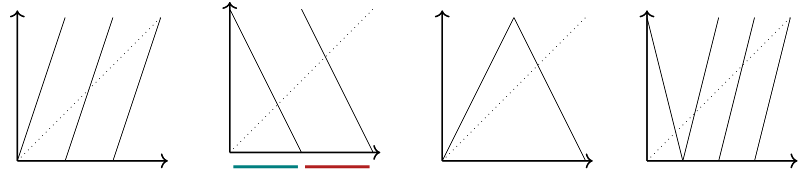
$$615243, 324156, 342516, 162534, 453621, 435261.$$

The smallest forbidden patterns of $G_4(x) = -4x \bmod 1$ are

$$123456, 654321, 123465, 654312.$$

Signed Shifts

Theorem (Elizalde & M.): $\pi \in \text{Allow}(\Sigma_\sigma)$ if and only if $\hat{\pi}^\star$ has the same monotonicity as σ and π is not "σ-collapsed."



$\sigma = +++$ $F_3(x) = 3x \mod 1$ | $\sigma = --$ $G_2(x) = -2x \mod 1$ | $\sigma = +-$ Tent Map | $\sigma = -+++$

Example: (Collapsed) Consider $G_2(x) = -2x \mod 1$, i.e. Σ_{--}.

$$\pi = 15423 \longrightarrow \hat{\pi}^\star = (\star, 5, 4, 2, 3) = \mathbf{53\star 24}.$$

Itinerary must begin **0100**, but the ending must satisfy

$$\mathbf{0}w_{[5,\infty)} <_{\text{alt}} w_{[5,\infty)} <_{\text{alt}} \mathbf{00}w_{[5,\infty)} \quad \rightsquigarrow \quad w_{[5,\infty)} = \mathbf{0}^\infty.$$

Pattern of length 5 for $\mathbf{01000}^\infty \leftrightarrow \frac{7}{12}$ not defined:

$$(x, G_2(x), \ldots G_2^4(x)) = \left(\frac{7}{12}, \frac{5}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$