A flag variety for the Delta Conjecture

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Outline

1. Permutations
    - algebra
    - geometry
2. Ordered Set Partitions
    - algebra
    - geometry
3. Delta Conjecture

Symmetric polynomials

S_n acts on $\mathbb{Q}[\mathbf{x}_n] := \mathbb{Q}[x_1, \ldots, x_n]$ by *permuting variables*.

$\mathbb{Q}[\mathbf{x}_n]^{S_n} = \{\text{symmetric polynomials in } x_1, \ldots, x_n\}$.

$$e_d = \sum_{1 \le i_1 < \cdots < i_d \le n} x_{i_1} \cdots x_{i_d}.$$

Thm: [Newton] $\{e_1, e_2, \ldots, e_n\}$ is an algebraically independent generating set of $\mathbb{Q}[\mathbf{x}_n]^{S_n}$.

Coinvariant algebra

S_n acts on $\mathbb{Q}[\mathbf{x}_n] := \mathbb{Q}[x_1, \ldots, x_n]$ by *permuting variables*.

The *invariant ideal* $I_n \subseteq \mathbb{Q}[\mathbf{x}_n]$ is

$$\begin{aligned} I_n &:= \langle \mathbb{Q}[\mathbf{x}_n]_+^{S_n} \rangle \\ &= \langle e_1, e_2, \ldots, e_n \rangle \end{aligned}$$

The *coinvariant algebra* is $R_n := \mathbb{Q}[\mathbf{x}_n]/I_n$.

Thm: [Chevalley] We have

$$R_n \cong \mathbb{Q}[S_n].$$

Graded Frobenius Image

Thm: [Chevalley] We have $R_n \cong \mathbb{Q}[S_n]$ as *ungraded* S_n-modules.

Q: What about the *graded* isomorphism type?

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 & 8 \\ \cline{1-3} 7 \\ \cline{1-1} \end{array}$$

$$\mathrm{maj}(T) = 3 + 5 + 6 = 14.$$
$$\mathrm{des}(T) = 3.$$

Graded Frobenius Image

Q: Recall $R_n \cong \mathbb{Q}[S_n]$. What about the *graded* isomorphism type?

Thm: (Lusztig-Stanley) We have

$$\mathrm{grFrob}(R_n; q) = \sum_{T \in \mathrm{SYT}(n)} q^{\mathrm{maj}(T)} \cdot s_{\mathrm{sh}(T)}(\mathbf{x}).$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

$$\mathrm{grFrob}(R_3; q) = q^0 \cdot s_{(3)}(\mathbf{x}) + q^1 \cdot s_{(2,1)}(\mathbf{x}) + q^2 \cdot s_{(2,1)}(\mathbf{x}) + q^3 \cdot s_{(1,1,1)}(\mathbf{x}).$$

Flag Variety

$$\begin{aligned}\mathcal{F}\ell(n) &= \{0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n \,:\, \dim(V_i) = i\} \\ &= GL_n/B\end{aligned}$$

Thm: [Ehresmann] The *Schubert cells* $\{X_w \,:\, w \in S_n\}$ where

$$X(w) = BwB/B$$

give a CW decomposition of $\mathcal{F}\ell(n)$.

Thm: [Borel] We have $H^\bullet(F\ell(n)) \cong R_n^{\mathbb{Z}}$ via $x_i \leftrightarrow -c_1(V_i/V_{i-1})$.
(Even as S_n-modules via *Springer action*.)

Ordered Set Partitions

Def: An *ordered set partition* is set partition of $[n]$ with a total order on its blocks.

Ex:

$$(135 \mid 6 \mid 24)$$

is an ordered set partition of $[6]$ with 3 blocks.

$$\mathcal{OP}_{n,k} := \{\text{all ordered set partitions } \sigma \models [n] \text{ with } k \text{ blocks}\}.$$

$$|\mathcal{OP}_{n,k}| = k! \cdot \mathrm{Stir}(n,k).$$

(No nice product formula.)

Q: Is there a nice quotient of $\mathbb{Q}[\mathbf{x}_n]$ reflecting the combinatorics of $\mathcal{OP}_{n,k}$?

New Generalized Coinvariant Algebra

Defn: [Haglund-R-Shimozono] For $k \leq n$, $I_{n,k} \subseteq \mathbb{Q}[\mathbf{x}_n]$ is the ideal

$$I_{n,k} := \langle e_n, e_{n-1}, \ldots, e_{n-k+1}, x_1^k, x_2^k, \ldots, x_n^k \rangle.$$

The ring $R_{n,k}$ is the corresponding quotient.

$$R_{n,k} = \mathbb{Q}[\mathbf{x}_n]/I_{n,k}$$

- $R_{n,k}$ is a graded $\mathfrak{S}_n$-module.
- $R_{n,1} = \frac{\mathbb{Q}[\mathbf{x}_n]}{\langle x_1, x_2, \ldots, x_n \rangle} \cong \mathbb{Q}$.
- $I_{n,n} = I_n$ and $R_{n,n} = R_n$.

Thm: [Haglund-R-Shimozono] $R_{n,k} \cong \mathbb{Q}[\mathcal{OP}_{n,k}]$ as ungraded S_n-modules. (Graded structure?)

Graded structure of $R_{n,k}$

Q: What about the *graded* S_n-module structure?

Thm: [Haglund-R-Shimozono]

$$\mathrm{grFrob}(R_{n,k}; q) = \sum_{T \in \mathrm{SYT}(n)} q^{\mathrm{maj}(T)} \begin{bmatrix} n - \mathrm{des}(T) - 1 \\ n - k \end{bmatrix}_q s_{\mathrm{sh}(T)}(\mathbf{x}).$$

Rmk: Proof uses an RSK argument due to Wilson.

Rmk: Meyer has an Adin-Brenti-Roichman style refinement of this result.

Rmk: Benkart-Colmenarejo-Harris-Orellana-Panova-Schilling-Yip have a crystal theoretic interpretation.

Geometric Wishful Thinking

Q: Is there a variety $X_{n,k}$ with $H^{\bullet}(X_{n,k}) \cong R_{n,k}^{\mathbb{Z}}$ as graded S_n-modules?

- When $k = n$, the flag manifold $\mathcal{F}\ell(n)$ works with Springer action.
- When $k = 1$, can take $X_{n,1} = \{*\}$.

$\mathrm{Hilb}(R_{n,k}; q)$ is not always palindromic:

$$\mathrm{Hilb}(R_{3,2}) = 1 + 3q + 2q^2$$

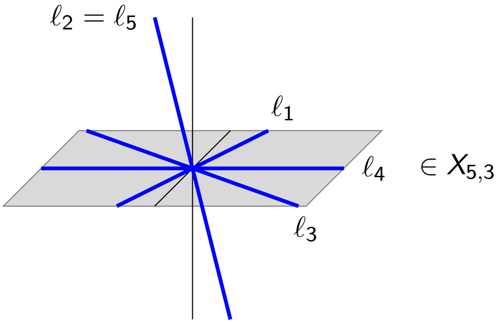
$\Rightarrow X_{n,k}$ cannot be a compact complex manifold.

Wish Granted!

Def: For $k \leq n$, let $X_{n,k}$ be the space of *line configurations*

$$X_{n,k} := \{(\ell_1, \ldots, \ell_n) : \ell_i \text{ a line in } \mathbb{C}^k \text{ and } \ell_1 + \cdots + \ell_n = \mathbb{C}^k\}.$$

S_n acts by permuting lines.



Thm: [PR] $H^\bullet(X_{n,k}) = R^{\mathbb{Z}}_{n,k}$ via $x_i \leftrightarrow -c_1(\ell_i)$ as graded rings or S_n-modules.

Representation Stability

Def: For $k \leq n$, let $X_{n,k}$ be the space of *line configurations*

$$X_{n,k} := \{(\ell_1, \ldots, \ell_n) : \ell_i \text{ a line in } \mathbb{C}^k \text{ and } \ell_1 + \cdots + \ell_n = \mathbb{C}^k\}.$$

We have S_n-equivariant embeddings

$$X_{n,k} \hookrightarrow X_{n+1,k}$$
$$X_{n,k} \hookrightarrow X_{n+1,k+1}$$

Fact: [PR] The towers

$$H^\bullet(X_{n,k}) \hookrightarrow H^\bullet(X_{n+1,k})$$
$$H^\bullet(X_{n,k}) \hookrightarrow H^\bullet(X_{n+1,k+1})$$

exhibit representation stability. (Geometric proof?)

Structure of $X_{n,k}$

Def: [PR] For $k \leq n$, let $X_{n,k}$ be the space of 'line configurations'

$$X_{n,k} = \{(\ell_1, \ldots, \ell_n) \,:\, \ell_i \text{ a line in } \mathbb{C}^k \text{ and } \ell_1 + \cdots + \ell_n = \mathbb{C}^k\}.$$

Thm: [PR] $X_{n,k}$ has a *paving by affines* with cells C_w indexed by *Fubini words* $w = w_1 \ldots w_n \in [k]^n$ (i.e., all the letters $1, 2, \ldots, k$ appear in w).

Ex: $(n,k) = (7,3)$ and $w = 2331231$:

$$C_w = \begin{pmatrix} 1 & 0 & 0 \\ \star & 1 & 0 \\ \star & \star & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & \star & \star & 0 & 1 & \star & \star \\ 0 & 1 & 1 & 0 & 0 & 1 & \star \end{pmatrix}.$$

Word Schubert Polynomials

Q: If $w = w_1 \ldots w_n \in [k]^n$ is a Fubini word, what is the class $[\overline{C}_w] \in H^\bullet(X_{n,k}) = R^{\mathbb{Z}}_{n,k}$?

$$\begin{aligned} w &= 2331231 \in [3]^7 \\ \mathrm{conv}(w) &= 2233311 \\ \mathrm{st}(\mathrm{conv}(w)) &= 2435617 \in S_7 \\ \sigma(w) &= 1523647 \in S_7 \end{aligned}$$

Thm: [PR] The class $[\overline{C}_w]$ is represented by

$$\mathfrak{S}_w := \sigma(w)^{-1}.\mathfrak{S}_{\mathrm{st}(\mathrm{conv}(w))} \in \mathbb{Z}[\mathbf{x}_n].$$

Cor: [PR] $\{\mathfrak{S}_w : w \in [k]^n \text{ Fubini}\}$ descends to a basis for $R^{\mathbb{Z}}_{n,k}$.

The Diagonal Coinvariants

Let S_n act on $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ *diagonally*:

$$w.x_i := x_{w_i}$$
$$w.y_i := y_{w_i}.$$

Def: The *diagonal coinvariant module* is the bigraded S_n-representation

$$DR_n := \mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n] / \langle \mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n]_+^{S_n} \rangle.$$

Thm: [Haiman] We have $\dim(DR_n) = (n+1)^{n-1}$. In fact (up to sign twist), DR_n is isomorphic to the permutation action of S_n on size n parking functions.

Bigraded Characters

Q: What is the bigraded S_n-module structure of DR_n?

Thm: [Haiman] The bigraded Frobenius series of DR_n is $\nabla(e_n)$, where ∇ is the Bergeron-Garsia nabla operator on symmetric functions (a *Macdonald eigenoperator*).

Problem: Expand $\nabla(e_n)$ in the *Schur basis* $\{s_\lambda : \lambda \vdash n\}$:

$$\nabla(e_n) = \sum_{\lambda \vdash n} c_\lambda(q,t) s_\lambda.$$

When λ is a *hook*, Haglund's q,t-Schröder Theorem gives an answer. No *conjecture* in general.

The Shuffle Theorem

'If symmetric functions are too hard, work with quasisymmetric functions.'

Thm: [Carlsson-Mellit] ('Shuffle Theorem') We have that

$$\nabla(e_n) = \sum_{P \in \mathrm{Park}_n} q^{\mathrm{area}(P)} t^{\mathrm{dinv}(P)} F_{\mathrm{iDes}(P)}.$$

The *Delta Conjecture* is a (conjectural) generalization of the Shuffle Theorem.

Delta Operators

- Λ_n = symmetric functions in $(x_1, x_2, \dots)$ of degree n.
- $\{\widetilde{H}_\mu : \mu \vdash n\}$ = modified Macdonald basis.
- $f = f(x_1, x_2, \dots)$ a symmetric function.

Def: $\Delta'_f : \Lambda_n \to \Lambda_n$ is the Macdonald eigenoperator defined by

$$\Delta'_f : \widetilde{H}_\mu \mapsto f(\dots, q^i t^j, \dots)\widetilde{H}_\mu,$$

where (i, j) range over all cells $\neq (0, 0)$ of the Ferrers diagram of μ.
Ex: $\mu = (4, 2) \vdash 6$.

$\bullet$	q	q^2	q^3
t	qt		

$$\Delta'_f(\widetilde{H}_\mu) = f(q, q^2, q^3, t, qt)\widetilde{H}_\mu$$

Fact: $\Delta'_{e_{n-1}}(e_n) = \nabla(e_n)$.

Delta Conjecture

Conj: [Haglund-Remmel-Wilson] For any $k \leq n$ we have

$$\Delta'_{e_{k-1}}(e_n) = \{z^{n-k}\} \left[\sum_{P \in \mathcal{LD}_n} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)} \prod_{i : a_i(P) > a_{i-1}(P)} \left(1 + z/t^{a_i(P)}\right) x^P \right]$$

$$= \{z^{n-k}\} \left[\sum_{P \in \mathcal{LD}_n} q^{\mathrm{dinv}(P)} t^{\mathrm{area}(P)} \prod_{i \in \mathrm{Val}(P)} \left(1 + z/q^{d_i(P)+1}\right) x^P \right],$$

where $\{z^{n-k}\}$ extracts the coefficient of z^{n-k}.

Rmk: When $k = n$, this is the Shuffle Theorem.

Def: Let $\mathrm{Rise}_{n,k}(\mathbf{x}; q, t), \mathrm{Val}_{n,k}(\mathbf{x}; q, t)$ denote the two right-hand sides.

$R_{n,k}$ and the Delta Conjecture

Delta Conj: [Haglund-Remmel-Wilson] For all $k \leq n$,

$$\Delta'_{e_{k-1}}(e_n) = \mathrm{Rise}_{n,k}(\mathbf{x}; q, t)$$
$$= \mathrm{Val}_{n,k}(\mathbf{x}; q, t).$$

Thm: [Wilson, R, Haglund-Garsia-Remmel-Yoo, HRS] We have

$$(*)\ \ \Delta'_{e_{k-1}} e_n|_{t=0} = \Delta'_{e_{k-1}} e_n|_{q=0, t=q} =$$
$$\mathrm{Rise}_{n,k}(\mathbf{x}; q, 0) = \mathrm{Rise}_{n,k}(\mathbf{x}; 0, q) = \mathrm{Val}_{n,k}(\mathbf{x}; q, 0) = \mathrm{Val}_{n,k}(\mathbf{x}; 0, q).$$

Thm: [HRS] Let $C_{n,k}(\mathbf{x}; q)$ be the symmetric function $(*)$. We have

$$\mathrm{grFrob}(R_{n,k}; q) = (\mathrm{rev}_q \circ \omega) C_{n,k}(\mathbf{x}; q).$$

$X_{n,k}$ and the Delta Conjecture

Thm: [PR] $\mathrm{grFrob}(H^\bullet(X_{n,k}); q) = (\mathrm{rev}_q \circ \omega)\Delta'_{e_{k-1}} e_n|_{t=0}$.

"$X_{n,k}$ is the flag variety attached to $\Delta'_{e_{k-1}}$."

Rmk: We also have flag varieties attached to Δ_{s_ν} when $\nu = (r, 1^{n-1})$.

Q: Can we find flag varieties for Δ_{s_ν} or Δ'_{s_ν} for all partitions ν?

Thanks for listening!

arXiv reading list:

- G. Benkart, L. Colmenarejo, P. Harris, R. Orellana, G. Panova, A. Schilling, and M. Yip. A minimaj-preserving crystal on ordered multiset partitions. Accepted, *Adv. Appl. Math.*, 2017.
- J. Haglund, B. Rhoades, and M. Shimozono. Ordered set partitions, generalized coinvariant algebras, and the Delta Conjecture. Accepted, *Adv. Math.*, 2018.
- K. Meyer. Descent representations of generalized coinvariant algebras. Preprint, 2017.
- B. Pawlowski and B. Rhoades. A flag variety for the Delta Conjecture. Preprint, 2017.