# $R$-systems

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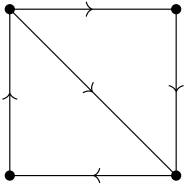
Joint work with Pavlo Pylyavskyy

# Part 1: Definition

# A system of equations

Let $G = (V, E)$ be a *strongly connected digraph*.



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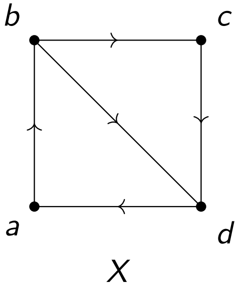

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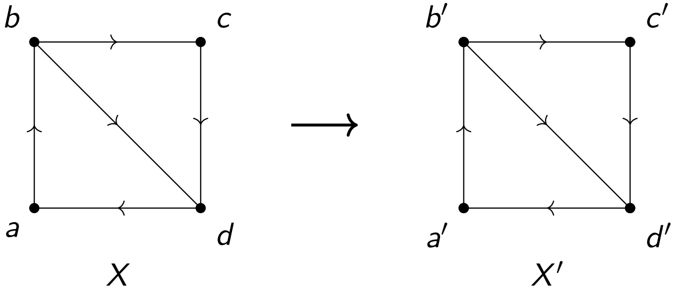

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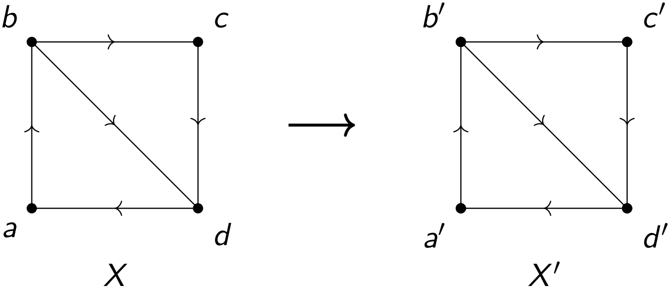


$$\forall\, v \in V, \quad X_v X'_v = \left( \sum_{v \to w} X_w \right) \left( \sum_{u \to v} \frac{1}{X'_u} \right)^{-1}$$

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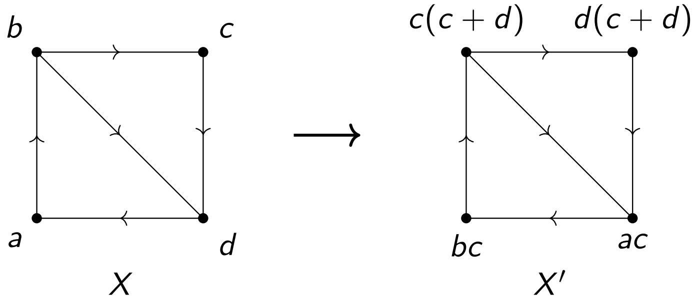


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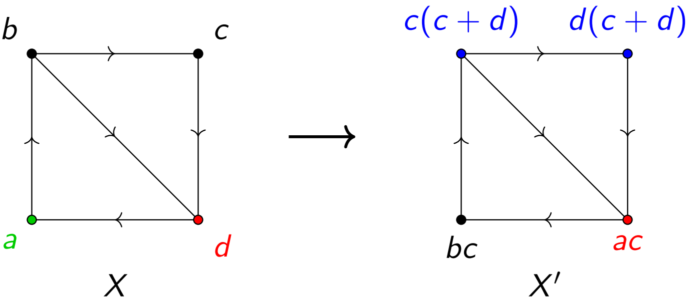


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$$d \cdot ac = (a)\left(\frac{1}{c(c+d)} + \frac{1}{d(c+d)}\right)^{-1}$$

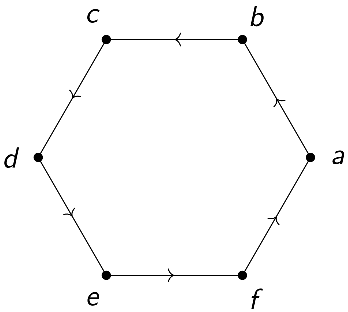
# Solution

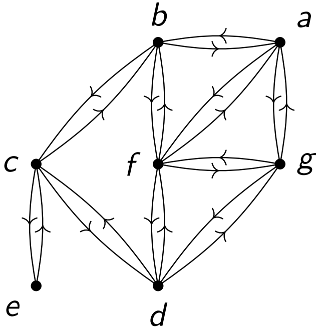
**Theorem (G.-Pylyavskyy, 2017)**

*Let $G = (V, E)$ be a strongly connected digraph. Then there exists a birational map $\phi : \mathbb{P}^V \dashrightarrow \mathbb{P}^V$ such that*

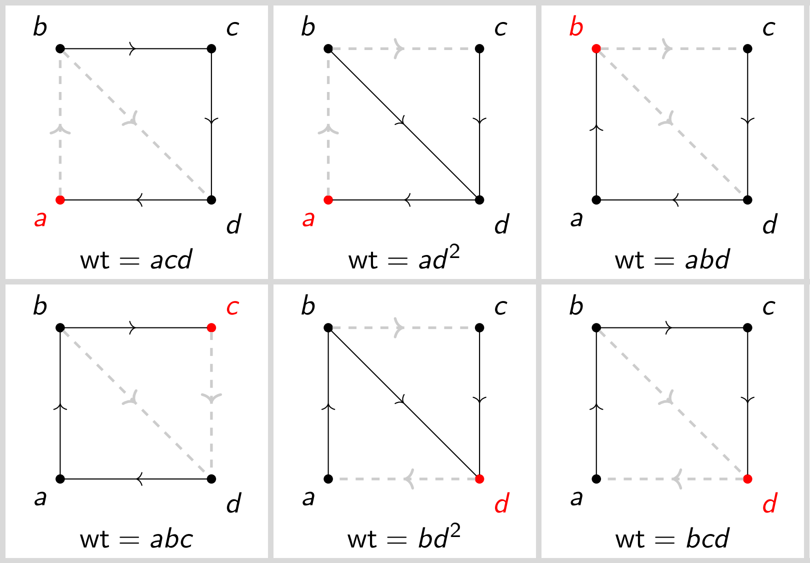
$$X, X' \in \mathbb{P}^V \text{ give a solution} \quad \Longleftrightarrow \quad X' = \phi(X).$$

# Periodic examples (exercise)

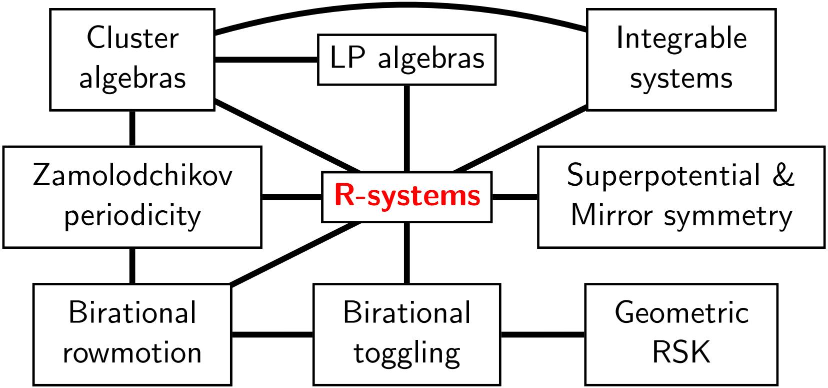


# Arborescence formula



# Map



Cluster algebras
LP algebras
Integrable systems
Zamolodchikov periodicity
R-systems
Superpotential & Mirror symmetry
Birational rowmotion
Birational toggling
Geometric RSK

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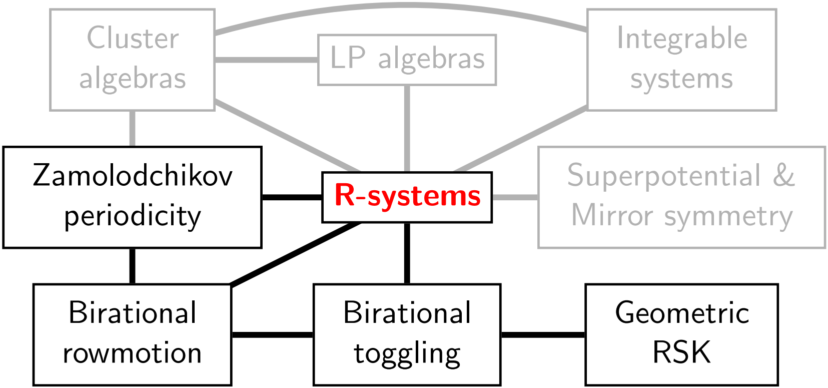

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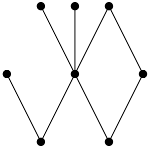
# Birational rowmotion ⊆ $R$-systems



$P$

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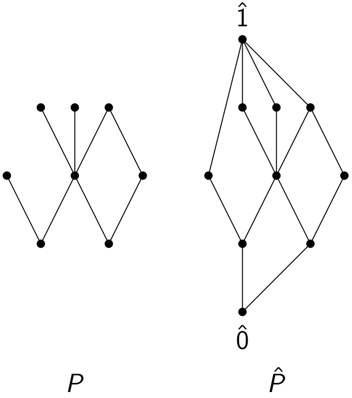

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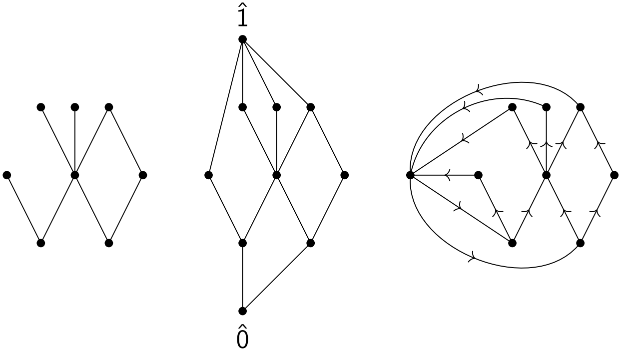


$P$ $\hat{P}$ $G(P)$

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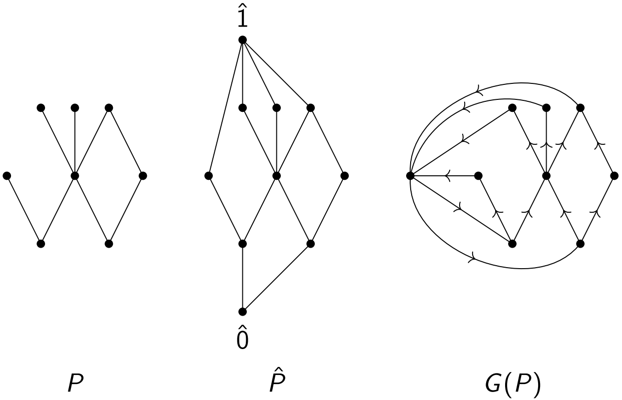
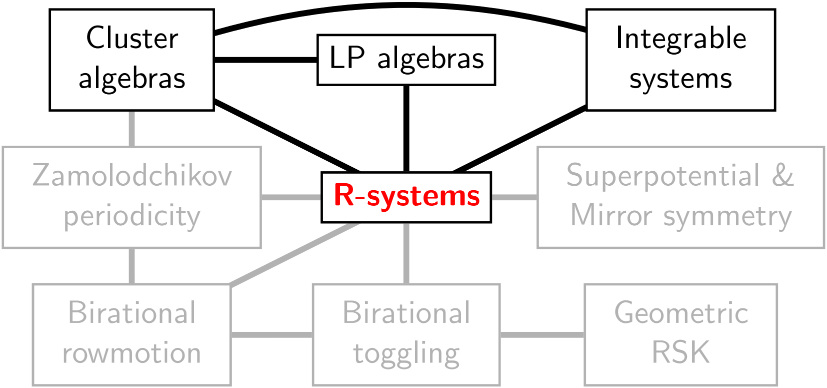



**Proposition (G.-Pylyavskyy, 2017)**

*Birational rowmotion on $P$ = $R$-system associated with $G(P)$.*

# Part 2: Singularity confinement

# Map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*For each $n > 4$, $\tau_n$ is a Laurent polynomial in $\alpha, \beta, \tau_1, \tau_2, \tau_3, \tau_4$.*

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---

A006720 Somos-4 sequence: a(0)=a(1)=a(2)=a(3)=1; for n >= 4, a(n) = (a(n-1) * a(n-3) + a(n-2)^2) / a(n-4). [78]
(Formerly M0857)

1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, 83313, 620297, 7869898, 126742987,
1687054711, 47301104551, 1123424582771, 32606721084786, 1662315215971057,
61958046554226593, 4257998884448335457, 334806306946199122193, 23385756731869683322514,
3416372868727801226636179 (list; graph; refs; listen; history; text; internal format)

# Singularity confinement

Consider a mapping of the plane $(x_{n-1}, x_n) \mapsto (x_n, x_{n+1})$ given by

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substitute $x_n = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}$

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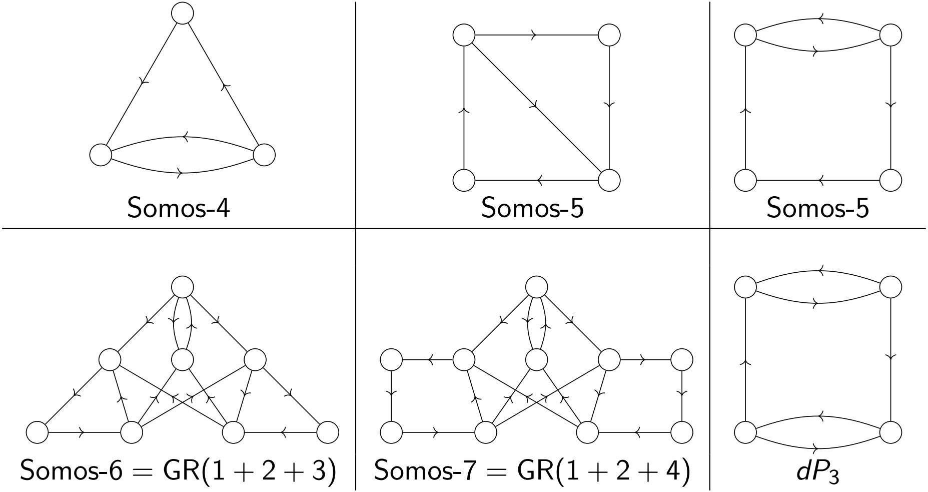
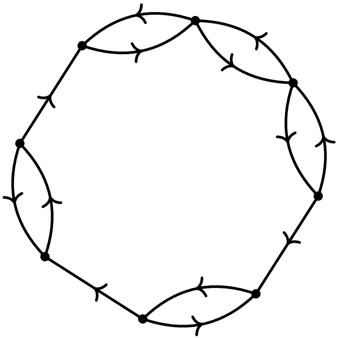
# Examples: Somos and Gale-Robinson sequences



# Examples: subgraphs of a bidirected cycle



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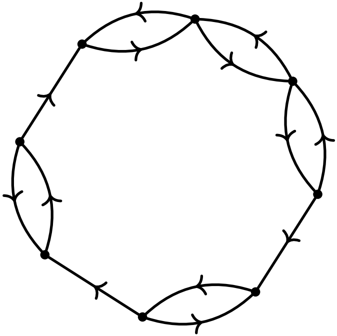
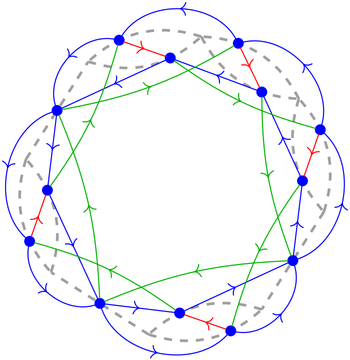
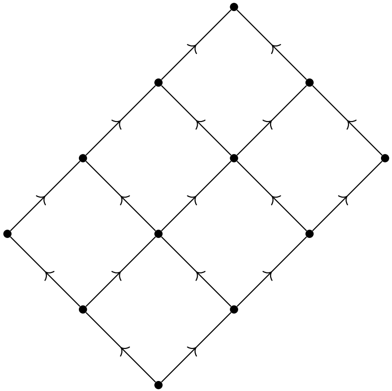




**Controlled by a cluster algebra**

# Examples: rectangle posets (Grinberg-Roby)



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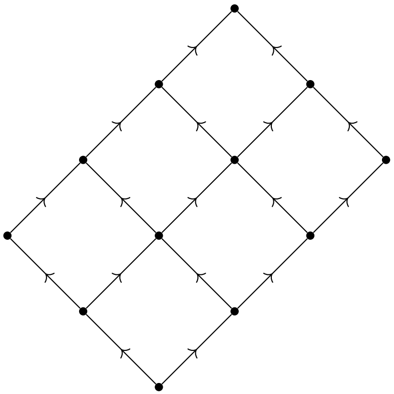
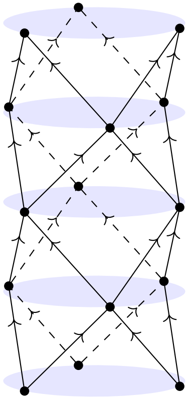


**Controlled by a $Y$-system**

# Examples: cylindric posets



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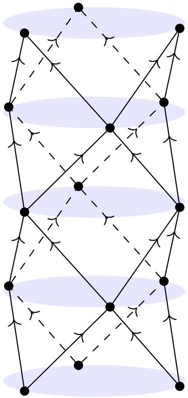
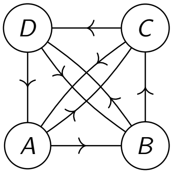
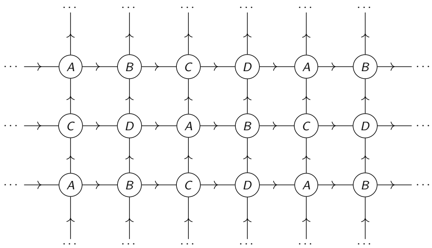


**Controlled by an LP algebra**

# Examples: toric digraphs



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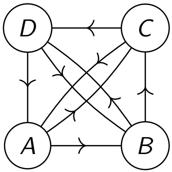


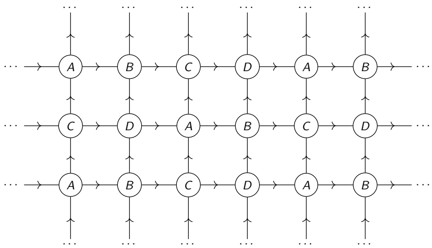


**Controlled by ???**

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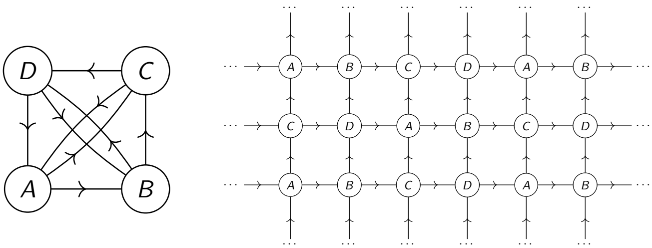


**Controlled by ???** $R_v(t) = \frac{\tau_v(t-1)}{\tau_v(t)}$;

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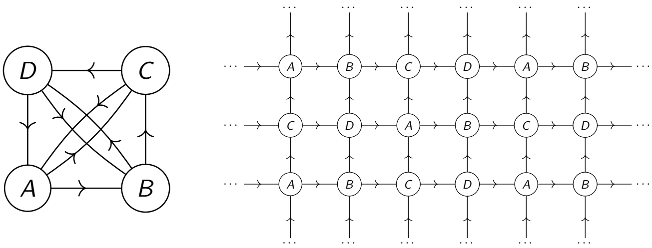


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## Conjecture (G.-Pylyavskyy, 2017)

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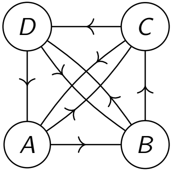


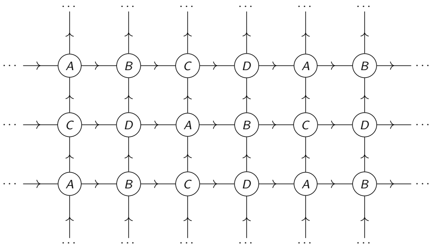


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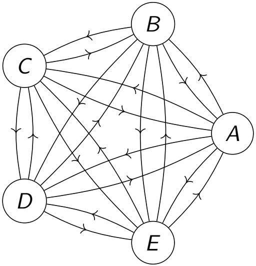
$$\tau_v(t+1) = \frac{\sum_{T \in \operatorname{Arb}(G;v)} \text{some product of } \tau_u(t)\text{-s and } \tau_w(t-1)\text{-s}}{\text{some other product of } \tau_u(t)\text{-s and } \tau_w(t-1)\text{-s}}.$$

# Example: the universal $R$-system



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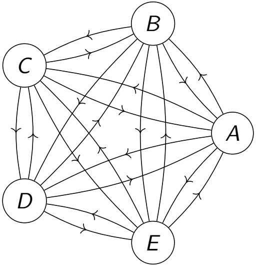


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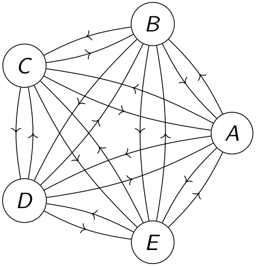


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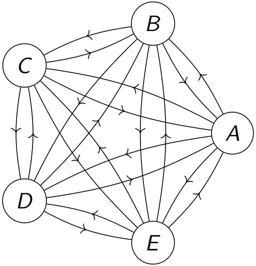


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**Conjecture (G.-Pylyavskyy, 2017)**

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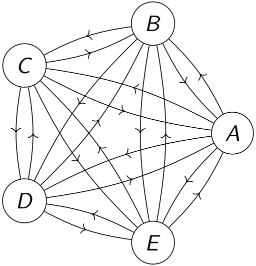


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# Bibliography

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Sage code: `http://math.mit.edu/~galashin/applets.html`


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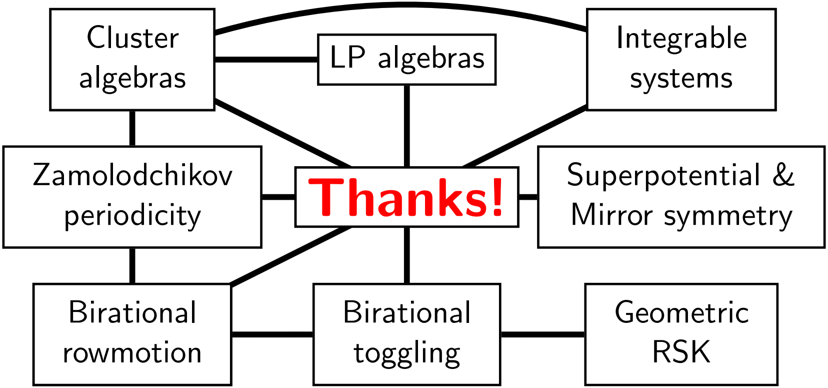

Cluster algebras
LP algebras
Integrable systems
Zamolodchikov periodicity
Thanks!
Superpotential & Mirror symmetry
Birational rowmotion
Birational toggling
Geometric RSK

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