Rowmotion on trim lattices

Hugh Thomas (UQAM) and Nathan Williams (UT Dallas)

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Rowmotion

Let P be a poset.

Let $\mathcal{L}(P)$ be the set of order ideals of P.

For $I \in \mathcal{L}(P)$, define rowmotion by

$$\rho(I) = P \setminus \{a \mid \exists b \in \max I, a \geq b\}$$

This defines a permutation of the set of order ideals of P.

Defining rowmotion via distributive lattices

$\mathcal{L}(P)$ is a distributive lattice and every distributive lattice is of the form $\mathcal{L}(P)$ for some P.

For $I \lessdot J$ in $\mathcal{L}(P)$, define $\gamma(J \gtrdot I) = J \setminus I$. This defines an edge-labelling of the Hasse diagram of $\mathcal{L}(P)$.

For $J \in \mathcal{L}(P)$, write $D(J) = \{\gamma(J \gtrdot I)\}$, the labels on the edges going down from J.

$$D(J) = \max J.$$

Write $U(J) = \{\gamma(I \gtrdot J)\}$, the labels on the edges going up from J.

$$U(J) = \min P \setminus J$$

Then ρ can be defined by $D(J) = U(\rho(J))$.

Another way to define rowmotion

For $a \in P$, and $I \in \mathcal{L}(P)$, define $\text{flip}_a(I)$ to be the element of $\mathcal{L}(P)$ obtained by walking along the edge of the Hasse diagram of P starting at I and labelled by a, if there is one.

Theorem (Cameron and Fon-der-Flaass)

Choose a linear extension of $P = a_1, \ldots, a_n$. Then $\rho(I) = \text{flip}_{a_n} \ldots \text{flip}_{a_1}(I)$.

This will be familiar to many of us at the *toggle definition* of rowmotion.

Descriptive labellings

We want to replace $\mathcal{L}(P)$ by some more general lattice $\mathcal{L}$.
Let $\gamma : \mathrm{cov}(\mathcal{L}) \to P$ be some labelling of the edges of $\mathcal{L}$.

We say that γ is *descriptive* if

- For $x \in \mathcal{L}$, all edges incident to x have different labels.
- x in $\mathcal{L}$ can be determined from knowing $D(x)$.
- x in $\mathcal{L}$ can be determined from knowing $U(x)$.
- $\{D(x) \mid x \in \mathcal{L}\} = \{U(x) \mid x \in \mathcal{L}\}$.

The labelling we have already defined of distributive lattices is descriptive.

If γ is a descriptive labelling, we can define a rowmotion as before by $D(x) = U(\rho(x))$. This is again a permutation of $\mathcal{L}$.

A more general example of a descriptive labelling

A lattice $\mathcal{L}$ is called semidistributive if $x \vee y = x \vee z$ implies both are equal to $x \vee (y \wedge z)$ and the dual assertion also holds.

Semidistributive lattices have a natural labelling of their Hasse diagram by join-irreducibles.

Emily Barnard showed that if $\mathcal{L}$ is semidistributive, then the labelling described above is descriptive (and thus defines a rowmotion).

Rowmotion in slow motion

Given a lattice and a descriptive labelling $\gamma : \text{cov}(\mathcal{L}) \to P$, we can define $\text{flip}_a(I)$ for $I \in \mathcal{L}$ as before.

We say that *rowmotion can be computed in slow motion* if for $x \in \mathcal{L}$, and any linear extension $P = \{a_1, \ldots, a_n\}$, we have

$$\rho(x) = \text{flip}_{a_n} \ldots \text{flip}_{a_1}(x).$$

For $\mathcal{L}$ a semidistributive lattice, the set of labels do not necessarily have an order with respect to which rowmotion can be computed in slow motion.

Main theorem

A lattice is called *trim* if it is *extremal* and *left modular*. Trim lattices include distributive lattices, finite Cambrian lattices, and intervals in them (so also Bergeron's m-Tamari lattices).

Left modular lattices have a natural edge-labelling.

Theorem

For $\mathcal{L}$ a trim lattice, this labelling is descriptive, and thus defines a rowmotion, which can be computed in slow motion.

Not all semidistributive lattices are trim, and not all trim lattices are semidistributive. This result is thus complementary to Barnard's.

Extremal lattices

The *length* of a lattice $\mathcal{L}$ is the length of the longest chain in $\mathcal{L}$.

A lattice of length ℓ has at least ℓ join-irreducible elements, and at least ℓ meet-irreducible elements.

Markowsky calls a lattice *extremal* if equality holds in both these bounds.

$\mathcal{L}(P)$ is extremal, since the length of all maximal chains is $|P|$ and there are bijections from P to both the join-irreducible and meet-irreducible elements.

Extremal lattices have a representation theorem due to Markowsky which generalizes Birkhoff's FTFDL.

Left modular lattices

Let $\mathcal{L}$ be a lattice. For any $x \in \mathcal{L}$, and any $y \leq z$, the *modular inequality* holds:

$$(y \vee x) \wedge z \geq y \vee (x \wedge z)$$

If the modular inequality holds with equality for x and for all $y \leq z$, then x is said to be *left modular*.

A lattice is said to be *left modular* if it has a maximal chain of left modular elements. Such lattices were studied by Blass and Sagan.

Left modularity of a lattice is a weakened form of supersolvability which does not imply gradedness.

Left modular labellings

A lattice equipped with a maximal left modular chain

$$\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$$

has a natural edge-labelling which can be defined by

$$\gamma(y \lessdot z) = \min\{i \mid z \wedge (x_i \vee y) = z\}$$

This definition also admits other equivalent formulations. See work of Larry Liu; McNamara-T.

This labelling is not descriptive in general, but it is descriptive for extremal left modular (=trim) lattices, so defines a rowmotion.

Our main result, stated earlier, is that this rowmotion, like the classic rowmotion of order ideals of a poset, can be computed in slow motion.

Thank you!