Root system chip-firing

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Distinctions

The root system chip-firing game is related to (but different from):

- The chip-firing game of Björner, Lovasz, and Shor ('91) (or Abelian Sandpile Model studied by Bak, Tang, and Wiesenfeld ('87) and Dhar ('90))

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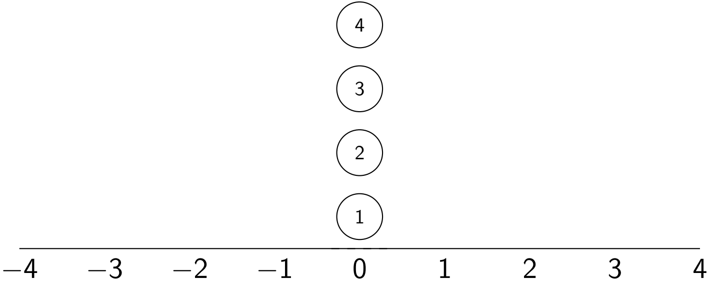
Root system chip-firing was developed as a generalization of Jim Propp's labeled chip-firing game.

Labeled chip-firing

- Among an infinite row of boxes, pick one box to place chips labeled $1, 2, \ldots, n$.



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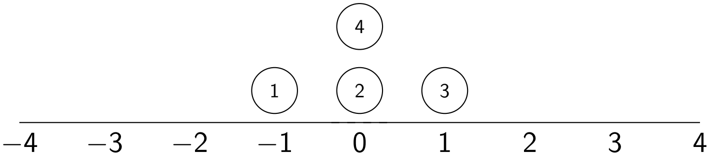

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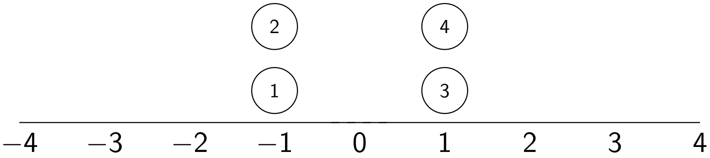

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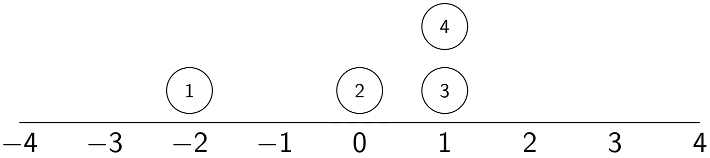

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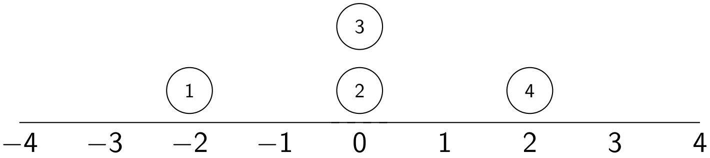

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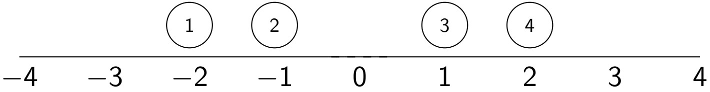

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Theorem (S Hopkins, T McConville, J Propp '17)

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Proof commentary: This is tricky. You are invited to develop a better proof.

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General root systems	Type A_{n-1}
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W Weyl group	$W = \mathfrak{S}_n$ permutations of $\{1,\dots,n\}$

Φ^+, P from last slide; $M \subseteq \mathbb{Z}$

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$\Phi^+ = \{e_j - e_i\}_{1 \leq i < j \leq n},\ P = \mathbb{Z}^n / \mathbb{Z}(1, \ldots, 1),\ M = \{1\}$

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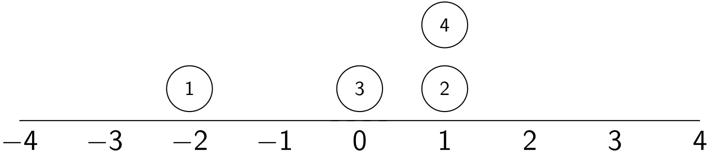

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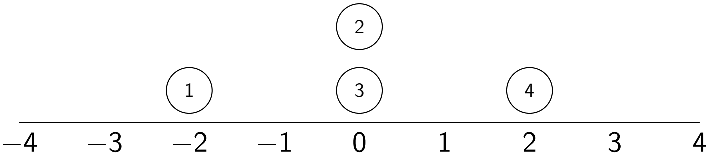

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Confluence

We say the root system chip-firing game is confluent from $\lambda_0 \in P$ if every sequence of moves $\lambda_0 \to \lambda_1 \to \lambda_2 \to \cdots$ terminates at the same weight μ. The game is confluent if it is confluent from every $\lambda_0 \in P$.

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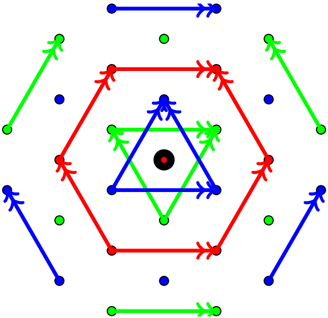
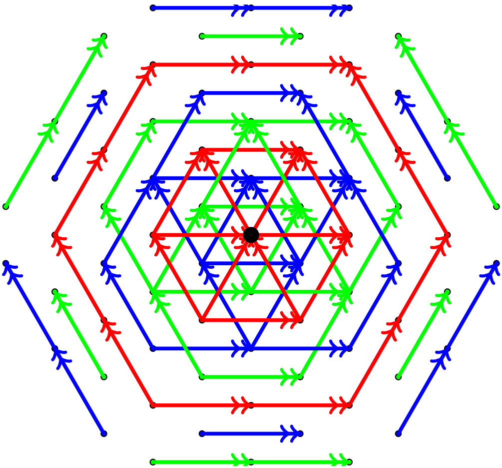
Theorem (P Galashin, S Hopkins, T McConville, A Postnikov '17)

For any root system Φ, the game is (fully) confluent when $M = \{-k, -k+1, \ldots, k-1, k\}$ or $M = \{-k+1, \ldots, k-1, k\}$ for some $k \in \{0, 1, 2, \ldots\}$.



Type A_2, $M = \{0\}$

Examples



Type A_2, $M = \{-1, 0, 1\}$

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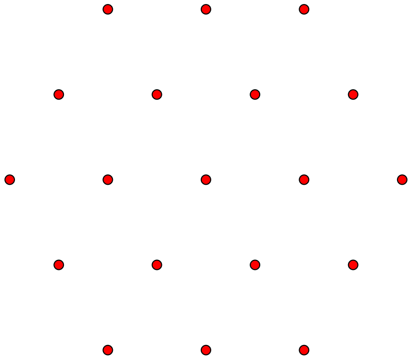
- $\mathrm{Perm}(\lambda)$ is the convex hull of $\{w \cdot \lambda | \; w \in W\}$
- $\Pi^Q(\lambda) = (Q + \lambda) \cap \mathrm{Perm}(\lambda)$



Non-escaping permutahedron lemma

- $P^+ = \{\lambda \in P | \langle \lambda, \alpha^\vee \rangle \geq 0, \ \forall \alpha \in \Phi^+\}$ dominant chamber
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If $\lambda \in P^+, \ \mu \in \Pi^Q(\lambda + k\rho)$ and $\mu \to \mu'$ then $\mu' \in \Pi^Q(\lambda + k\rho)$.

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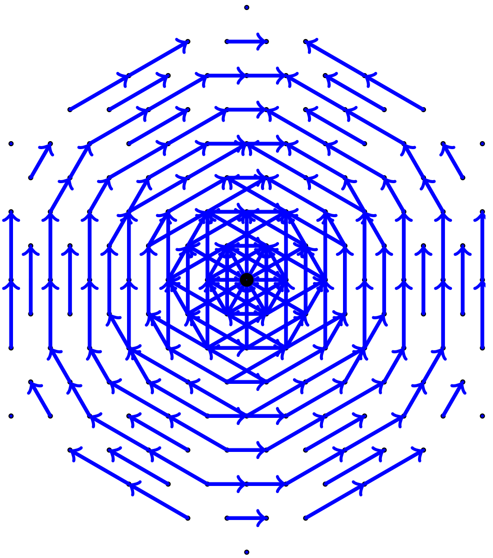
If $\lambda_0, \lambda \in P^+$ where λ is minimal in dominance order such that $\lambda_0 \in \Pi^Q(\lambda + k\rho)$, then $\lambda_0 \to \cdots \to \lambda + k\rho$.

Symmetric example



$G_2, \;\; M = \{-1, 0, 1\}$

Truncated intervals $M = \{-k+1, \ldots, k-1, k\}$

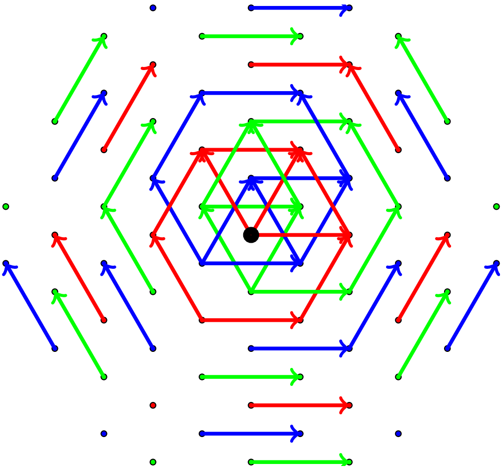
Theorem (GHMP)

For a truncated interval M, the game is (fully) confluent.

Proof.

Local confluence may be proved by reduction to rank 2. □

Truncated example



$$A_2, \;\; M = \{0, 1\}$$

Closing remarks

- Let $H_{\alpha,j} = \{x \in \mathbb{R}^n | \ \langle x, \alpha^\vee \rangle = j\}$. The affine hyperplane arrangements

$$\mathcal{A}_k^{\text{Cat}} = \{H_{\alpha,j} | \ \alpha \in \Phi^+, \ j \in \{-k, \dots, k\}\},$$

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  are the k-Catalan and k-Shi arrangements, respectively.
- There are polynomials $e_\lambda(k)$ indexed by dominant weights λ that count the set of λ_0 such that $\lambda_0 \to \cdots \to \lambda + k\rho$ (for the symmetric or truncated cases). This polynomial has positive integer coefficients (S Hopkins, GHMP). The $e_\lambda(k)$ like an Ehrhart polynomial for a (sometimes) non-convex region!

THANKS!!!