The Berenstein-Kirillov group and cactus groups

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January 12, 2018

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arxiv:1609.02046

Overview

- Berenstein-Kirillov group BK_n
  - generated by Bender Knuth involutions on SSYT
  - includes many classical operators e.g. promotion, evacuation, jdt, symmetric group action
- Cactus group C_n
  - arises as fundamental group of certain moduli space
  - relates to coboundary categories, Kazhdan-Lusztig thoery
  - acts on S_n by “interval evacuation”

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Theorem (Chmutov–G.–Pylyavskyy)

There is a surjective homomorphism from C_n to BK_n "compatible" with the respective actions of the groups on S_n and SSYT.

Outline

1. Definition of the groups
2. Main results
3. Proofs with growth diagrams

Bender Knuth involutions

- $SSYT$ = {semistandard Young tableaux of all shapes}
- $t_i =$ Bender Knuth involution ($t_i : SSYT \to SSYT$)

Example

$$t_3\left(\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&2&3&3&4\\\hline 2&2&3&3&4&4&4\\\cline{1-7} 3&3&4\\\cline{1-3}\end{array}\right)=\begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&2&3&3&4\\\hline 2&2&3&3&3&4&4\\\cline{1-7} 4&4&4\\\cline{1-3}\end{array}$$

Berenstein-Kirillov group [BK 1995]

- Combinatorial definition:

$$BK_n = \text{Subgroup of } \{\text{Bijections of } SSYT\} \\ \text{generated by } t_1, t_2, \ldots, t_{n-1}$$

- Abstract "definition":

$$BK_n = \langle t_1, \ldots, t_{n-1} \mid R_1, R_2, R_3, (\text{Possibly more}) \rangle$$

where

(R1) $t_i^2 = 1,\ t_i t_j = t_j t_i$ if $|i - j| \geq 2$
(R2) $(t_1 t_2)^6 = 1$
(R3) $(t_1 q_j)^4 = 1$ if $j \geq 3$
where $q_j = (t_1)(t_2 t_1) \cdots (t_j \cdots t_1)$

The cactus group [Henriques-Kamnitzer 2006]

$$C_n = \langle \tau_{i,j}, 1 \leq i < j \leq n \mid S_1, S_2, S_3 \rangle$$

where

(S1) $\tau_{i,j}^2 = 1$

(S2) $(\tau_{i,j}\tau_{k,l})^2 = 1$, if $i < j < k < l$

(S3) $\tau_{i,j}\tau_{k,l}\tau_{i,j} = \tau_{i+j-l,i+j-k}$, if $i < k < l < j$

Action of C_n on permutations

- Define $\tau_{i,j}(\pi_1 \cdots \pi_n) = \pi_1 \cdots \pi_i' \cdots \pi_j' \cdots \pi_n$ where

$$\pi_i' \cdots \pi_j' = \mathrm{RSK}^{-1} \circ (id \times \mathrm{Evac}) \circ \mathrm{RSK}(\pi_i \cdots \pi_j)$$

(Evac = evacuation $= q_{j-i}$).

Example

$\tau_{2,4}(15243) = 12543$ because

$$524 \longrightarrow \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 5 & \\ \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \end{array}$$

$$\downarrow$$

$$254 \longleftarrow \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 5 & \\ \end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \end{array}$$

Action of C_n on permutations (cont.)

- Define $\tau_{i,j}(\pi_1 \cdots \pi_n) = \pi_1 \cdots \pi_i' \cdots \pi_j' \cdots \pi_n$ where

$$\pi_i' \cdots \pi_j' = \mathrm{RSK}^{-1} \circ (id \times \mathrm{Evac}) \circ \mathrm{RSK}(\pi_i \cdots \pi_j).$$

Theorem (Losev 2015, Henriques-Kamnitzer 2006)

The $\tau_{i,j}$ satisfy the relations of the cactus group.

Cactus-like generators for BK_n

- Define $q_{[i,j]} = q_{j-1} q_{j-i} q_{j-1} \in BK_n$.
- Examples
  - $q_{[1,j]} = q_{j-1}^3 = q_{j-1}$
  - $q_{[j,j+1]} = q_j q_1 q_j = s_j$
  - $q_{[2,4]} = q_3 q_2 q_3 = (t_1 t_2 t_1 t_3 t_2 t_1)(t_1 t_2 t_1)(t_1 t_2 t_1 t_3 t_2 t_1)$

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Theorem (CGP)

The element $q_{[i,j]}$ acts locally on SSYT: if $T' = q_{[i,j]}(T)$ then

1. $T' = T$ *except on the interval* $\{i, i+1, \ldots, j\}$ *and*
2. $T'|_{\{i,i+1,\ldots,j\}}$ *depends only on* $T|_{\{i,i+1,\ldots,j\}}$.

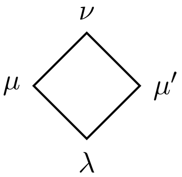
The homomorphism

Theorem (CGP)

1. *There is a surjective homomorphism* $C_n \to BK_n$ *taking* $\tau_{i,j}$ *to* $q_{[i,j]}$ *for each* $i < j$.
2. *For each* $\pi \in S_n$

$$\tau_{i,j}(\pi) = (RSK^{-1} \circ (id \times q_{[i,j]}) \circ RSK)(\pi)$$

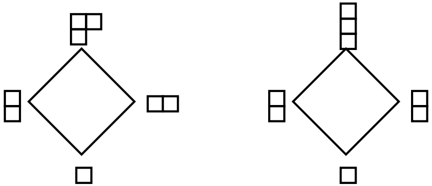
Fomin growth rule



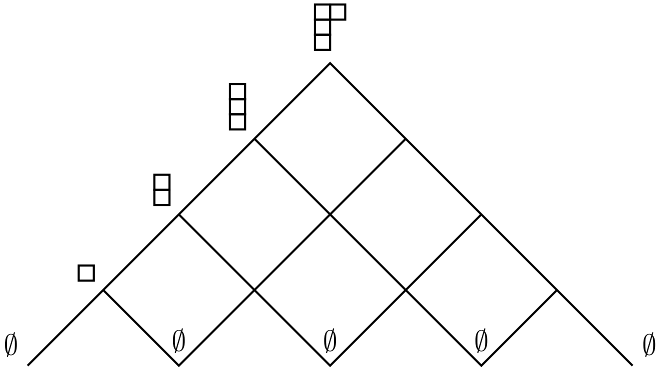
If $\lambda <_c \mu <_c \nu$ (cover relations in Young's lattice) then

$$\mu' = \begin{cases} \lambda \cup (\nu \setminus \mu), & \text{if it is a shape} \\ \mu, & \textit{otherwise} \end{cases}$$

Fomin's growth rule (examples)



Growth diagram that calculates evacuation



$$\mathrm{Evac}\left(\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right) =$$

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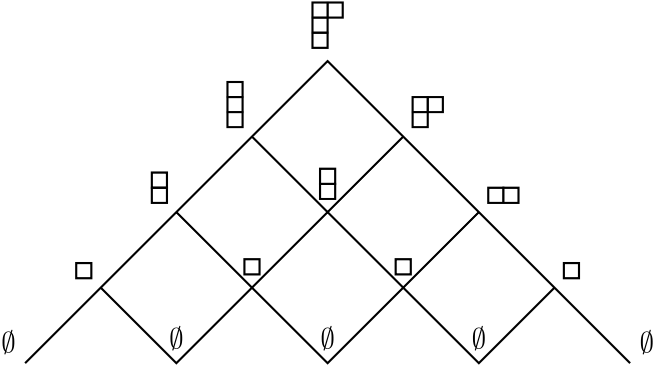

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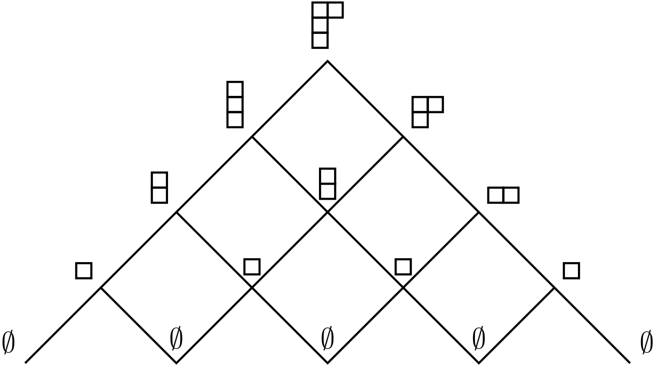


$$\mathrm{Evac}\left(\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}\right) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$$

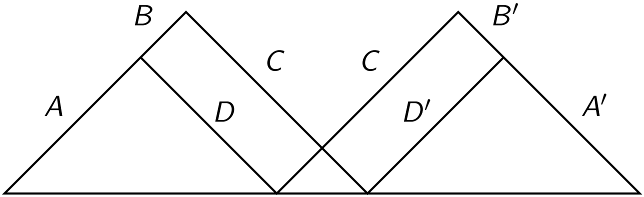
Growth diagrams

More generally, a growth diagram with diamonds at heights $i_1, i_2, \ldots, i_m$ calculates the element

$$t_{i_m} \cdots t_{i_2} t_{i_1} \in BK_n$$

Proof by picture of weak locality

Let $q_{[i,j]}(T) = T'$.



Need to show

$$A = T|_{\{1,\dots,i-1\}} = T'|_{\{1,\dots,i-1\}} = A'.$$