The coincidental down-degree expectations (CDE) property for posets and homomesy

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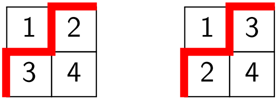
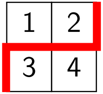
SYT-compatible lattice paths

Let T be an standard Young tableau of shape $a \times b$. We say a NE-lattice path from SW corner of $a \times b$ to NE corner is *compatible* with T if the entries above the path are all less than the entries below. For example, the following lattice path is compatible with both SYT of shape 2×2:

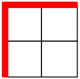
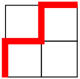
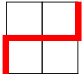
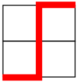
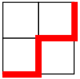
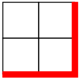


while the following lattice path is compatible with only one of these SYTs:



Let $\mu_{\max}$ denote the distribution on paths where each occurs proportional to the number of SYTs it is compatible with.

Expected number of turns

Lattice path p						
$\mu_{\max}(p)$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$
# of turns	1	3	2	2	3	1

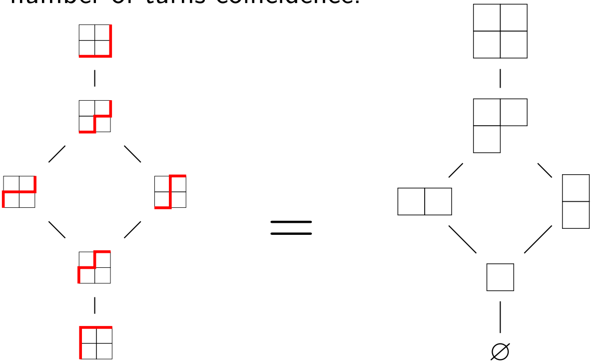
Theorem (Chan-López Martín-Pfleuger-Teixidor i Bigas, 2015)

$\mathbb{E}_{\mu_{\max}}(\#\ \textit{of turns})$ *for lattice paths in* $a \times b$ *is* $2ab/(a+b)$.

This was the key combinatorial result CLPT needed to reprove a genus formula for Brill-Noether curves due to Eisenbud-Harris and Pirola. CLPT note curiously that this is the same expected number of turns as for the *uniform* distribution μ_{uni} on paths...

Poset of lattice paths

Reiner, Tenner, and Yong (2016) gave a poset-theoretic reformulation of this expected number of turns coincidence:



This poset is the same as the interval $[\varnothing, b^a]$ in Young's lattice. Note:

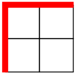
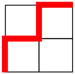
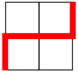
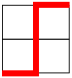
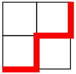
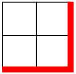
- SYTs correspond to maximal chains;
- # of turns = degree in the Hasse diagram;
- # of "leftwards" turns = down-degree in the Hasse diagram.

The CDE property

Let P be a (finite) poset. Let μ_{uni} denote the uniform distribution on P and let μ_{max} denote the distribution on P in which each element is weighted proportional to the number of maximal chains passing through it.

Definition (RTY, 2016)

P has the *coincidental down-degree expectations* property (or "P is CDE") if the expected down-degree is the same w.r.t μ_{max} as w.r.t. μ_{uni}.

$p \in P$						
$\mu_{\mathrm{max}}(p)$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$
$\mathrm{ddeg}(p)$	0	1	1	1	2	1

Basic CDE results of RTY

A chain $[m]$ is CDE because in this case $\mu_{\mathrm{max}} = \mu_{\mathrm{uni}}$. Here are some other basic results obtained by RTY:

Lemma (RTY, 2016)

Let P and Q be graded posets (= all maximal chains have same length). Suppose P and Q are CDE. Then $P \times Q$ is CDE.

Corollary (RTY, 2016)

All Boolean lattices are CDE.

Lemma (RTY, 2016)

Let P be self-dual of constant Hasse diagram degree. Then P is CDE.

Corollary (RTY, 2016)

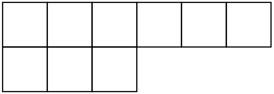
The weak order in any type is CDE. The Tamari lattice is CDE.

CDE intervals of Young's lattice: stretched staircases

Corollary (CLPT, 2015)

$[\varnothing, b^a]$ *is CDE with expectation* $(ab)/(a+b)$.

What about other intervals of Young's lattice? Let $\delta_n := (n, n-1, \ldots, 1)$ be the staircase partition, and let $\delta_n \circ b^a$ be the result of replacing each box of δ_n with a b^a rectangle; e.g. $\delta_2 \circ 3^1$ is:



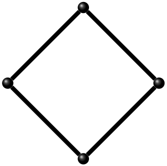
Theorem (RTY, 2016)

$[\varnothing, \delta_n \circ b^a]$ *is CDE with expectation* $(nab)/(a+b)$.

RTY's proof is via insertion algorithms. Are there other CDE intervals of Young's lattice? Actually, answer was already obtained previously by Chan, Haddadan, Hopkins, and Moci (CHHM, 2015)! Let's explain how...

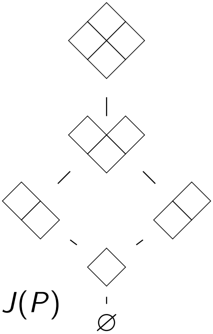
Order ideals and distributive lattices

Let P be a poset. An *order ideal* of P is a downwards-closed subset, i.e. a subset $I \subseteq P$ with $y \in I$ and $x \leq y$ implies $x \in I$. We use $J(P)$ to denote the set of order ideals of P. Actually, $J(P)$ is a poset where order ideals are ordered by containment.



$P = [2] \times [2]$

$\Rightarrow$



$J(P)$

A *distributive lattice* is a poset of form $J(P)$ for some (in fact, unique) P.

Toggle-symmetric distributions

If $I \in J(P)$ and $p \in P$, then *toggling at p* is the operation of

- adding p to I if $p \notin I$ and $I \cup \{p\}$ is still an order ideal; *(toggling in)*
- removing p from I if $p \in I$ and $I \setminus \{p\}$ is still an order; *(toggling out)*
- doing nothing otherwise.

Denote the result by $\tau_p(I)$. The *toggle-group* is the subgroup of $S_{J(P)}$ generated by the τ_p, introduced by Cameron and Fon-der-Flaass. Set

$$\mathcal{T}_p^+(I) := \begin{cases} 1 & \text{if } p \text{ can be toggled into } I; \\ 0 & \text{otherwise.} \end{cases}$$

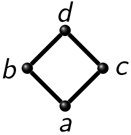
$$\mathcal{T}_p^-(I) := \begin{cases} 1 & \text{if } p \text{ can be toggled out of } I; \\ 0 & \text{otherwise.} \end{cases}$$

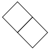
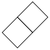
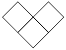
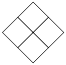
Definition (CHMM, 2015)

Let μ be a distribution on $J(P)$. We say μ is *toggle-symmetric* if for every element $p \in P$ we have $\mathbb{E}_\mu(\mathcal{T}_p^+) = \mathbb{E}_\mu(\mathcal{T}_p^-)$.

Toggle-symmetry: example

$P = [2] \times [2]$:



$I \in J(P)$	$\varnothing$					
$\mu_{\max}(I)$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$
$\mathcal{T}_b^+(I)$	0	1	0	1	0	0
$\mathcal{T}_b^-(I)$	0	0	1	0	1	0

So $\mathbb{E}_{\mu_{\max}}(\mathcal{T}_b^+) = \frac{3}{10} = \mathbb{E}_{\mu_{\max}}(\mathcal{T}_b^-)$. In fact, if we check for the other elements a, c, and d, we will find that $\mu_{\max}$ is toggle-symmetric.

The toggle CDE property

It is easy to see that μ_{uni} on $J(P)$ is always toggle-symmetric because I and $\tau_p(I)$ are equally probably w.r.t. μ_{uni} for all $p \in P$ and $I \in J(P)$.

Lemma (CHMM, 2015)

The distribution μ_{max} on $J(P)$ is always toggle-symmetric.

Definition (Hopkins, 2016)

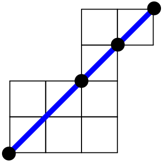
We say $J(P)$ is *toggle CDE* (tCDE) if the expected down-degree is the same w.r.t. all toggle-symmetric distributions μ on $J(P)$.

The above lemma says that $J(P)$ being tCDE implies $J(P)$ is CDE.
Here is a key observation for proving some $J(P)$ is tCDE:

$$\mathrm{ddeg}(I) = \#\mathrm{max}(I) = \sum_{p \in P} \mathcal{T}_p^-(I).$$

"Balanced" skew shapes

An *outward corner* of λ/ν is a corner where we "could add a box." E.g., with $\lambda/\nu = (4,3,3,3)/(2,2)$ the outward corners are marked with dots:



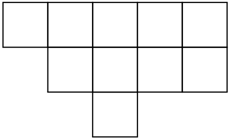
We have drawn the *main anti-diagonal* of λ/ν (= line connecting SW to NE corners) in blue here. We say λ/ν is *balanced* if all outward corners occur at the main anti-diagonal. Observe that $\delta_n \circ b^a$ is always balanced.

Theorem (CHHM, 2015)

Let λ/ν be a balanced skew shape of height (= # of rows) a and width (= # of columns) b. Then $[\nu, \lambda]$ is tCDE with expectation $(ab)/(a+b)$.

The shifted Young's lattice

A partition is *strict* if its parts strictly decrease in size; e.g. $(4,3,1)$ is strict but $(4,3,3)$ is not. Associated to a strict partition is its *shifted Young diagram* where we indent each row by one box; e.g. the shifted Young diagram of $\lambda = (5,4,1) = \delta_3 + 2^2$ is



The shifted Young's lattice is the restriction of Young's lattice to strict partitions. $[\varnothing, \lambda]_{\mathrm{shift}}$ denotes an interval of the shifted Young's lattice.

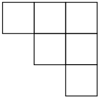
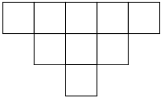
Conjecture (RTY, 2016)

For $ma < n$, $[\varnothing, \delta_n + \delta_m \circ a^a]_{\mathrm{shift}}$ is CDE with expectation $(n + ma + 1)/4$.

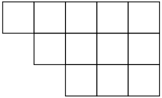
Shifted-balanced shapes

Let $0 \leq k < n$. Let ν be balanced of width and height both $= k$. Say λ is

- *shifted-balanced of Type (1)* if it is of form $\lambda = \delta_n + \nu$:

e.g.,  or 

- *shifted-balanced of Type (2)* if it is of form $\lambda = \delta_n + (n-1-k)^n + \nu$:

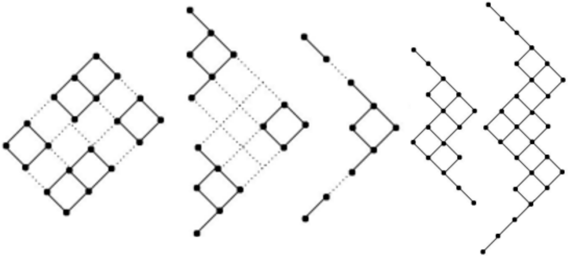
e.g., 

Theorem (Hopkins, 2016)

Let λ be shifted-balanced of Type (1) or (2). Then $[\varnothing, \lambda]_{\mathrm{shift}}$ is tCDE of expectation $(\lambda_1 + 1)/4$.

Minsucule lattices

The *minuscule posets* are certain posets arising in the representation theory of Lie algebras that enjoy many remarkable combinatorial properties (e.g. only examples of *Gaussian posets*). They have been classified:



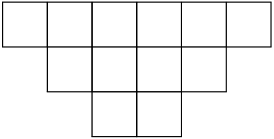
Corollary (CHHM, 2015 and Hopkins, 2016)

For any minuscule poset P, the associated lattice $J(P)$ is tCDE.

Result has subsequently been proven in a *uniform* manner by Rush.

Trapezoids

A shifted shape of the form $\lambda = (n, n-2, \ldots, n-2k)$ is a "trapezoid":



Conjecture (RTY, 2016)

The trapezoid $[\varnothing, (n, \ldots, n-2k)]_{\mathrm{shift}}$ *is CDE with expectation* $|\lambda|/(n+1)$.

For $k = 0$ or $k = (n-1)/2$ (with n odd) this reduces a previous result. But for all other k, it turns out that the trapezoid is **not tCDE** even though it is (apparently) CDE. This is somewhat surprising (to me). (I was able to prove the trapezoid conjecture for $k = 1$ "by hand.")

Rowmotion

Rowmotion is a certain well-studied action on order ideals of a poset; according to Cameron and Fon-der-Flaass (1993), we have

$$\Phi_{\mathrm{row}} := \tau_{p_1} \circ \tau_{p_2} \circ \cdots \circ \tau_{p_n}$$

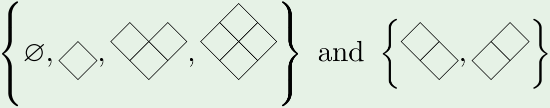
where $p_1, p_2, \ldots, p_n$ is any linear extension of P.

Theorem (Brouwer-Schrijver, 1974)

The order of Φ_{row} for $P = [a] \times [b]$ is $a + b$.

Example

Let $P = [2] \times [2]$. Then rowmotion has two orbits:



Homomesy of antichain cardinality

Propp and Roby (2015) introduced the *homomesy phenomenon*.

Definition (Propp-Roby, 2015)

Let $\mathcal{S}$ be a combinatorial set, $\varphi\colon \mathcal{S} \to \mathcal{S}$ some invertible map, and $f\colon \mathcal{S} \to \mathbb{R}$ a combinatorial statistic. We say that f is c-mesic with respect to the action of φ on $\mathcal{S}$ if the average of f along every φ-orbit is c.

One of their (canonical?) examples of homomesy is the following:

Theorem (Propp-Roby, 2015)

The statistic $\#\mathrm{max}(I)$ is $(ab)/(a+b)$-mesic with respect to the action of Φ_{row} on $J([a]\times[b])$.

Gyration and "rank-permuted" rowmotion

Suppose that P is ranked. *Gyration* is the toggle-group element

$$\Phi_{\mathrm{gyr}} := \tau_{p_{e_1}} \circ \tau_{p_{e_2}} \circ \cdots \circ \tau_{p_{e_k}} \circ \tau_{p_{o_1}} \circ \tau_{p_{o_2}} \circ \cdots \circ \tau_{p_{o_l}}$$

where $e_1, \ldots, e_k$ are the elements of even rank and $o_1, \ldots, o_l$ are the elements of odd rank. Gyration was originally introduced by Wieland (2000) in the study of Alternating Sign Matrices.

More generally, for any permutation σ of the ranks $0, 1, ..., r$ of P, define

$$\Phi^{\sigma}_{\mathrm{row}} := \text{toggle all of rank } \sigma(0) \circ \text{toggle rank } \sigma(1) \circ \cdots \circ \text{toggle rank } \sigma(r).$$

Cameron and Fon-der-Flaass (1993) showed that all the $\Phi^{\sigma}_{\mathrm{row}}$ are conjugate under the toggle-group.

Toggle-symmetry of rowmotion/gyration orbits

Striker (2015) showed what amounts to the following:

Theorem (Striker, 2015)

For any (ranked) P, the distribution that is constant on any Φ_{row}-orbit (or any Φ_{gyr}-orbit) $\mathcal{O} \subseteq J(P)$ is toggle-symmetric.

In fact, her arguments show more generally that:

Theorem

For any ranked P and any σ, the distribution that is constant on any $\Phi^{\sigma}_{\mathrm{row}}$-orbit $\mathcal{O} \subseteq J(P)$ is toggle-symmetric.

Antichain cardinality homomesy corollaries

Stiker's result lets us immediately obtain many homomesy results from our study of tCDE posets.

Corollary (CHHM, 2015)

Let λ/ν be a balanced skew shape of width a and height b. Then $\#\mathrm{max}(I)$ is $(ab)/(a+b)$-mesic w.r.t. the action of $\Phi^{\sigma}_{\mathrm{row}}$ on $[\nu,\lambda]$.

Corollary (Hopkins, 2016)

Let λ be shifted-balanced of Type (1) or (2). Then $\#\mathrm{max}(I)$ is $(\lambda_1+1)/4$-mesic w.r.t. the action of $\Phi^{\sigma}_{\mathrm{row}}$ on $[\varnothing,\lambda]_{\mathrm{shift}}$.

Observe that these generalize Propp-Roby's result and to my knowledge most of these results were not known previously.

A few stray thoughts...

- Based on my computations, it seems that balanced shapes are the only initial intervals of Young's lattice that are tCDE/CDE. Similar story with shifted shapes. Is this true?
- Rectangles, staircases, "double shifted staircases," and trapezoids are known or conjectured to be CDE. These are also essentially the only posets where order of promotion is understood; see *"Dual equivalence with applications..."* by Haiman, 1992. Is there some connection here? Note that promotion orbits are **not** toggle-symmetric.
- Does homomesy for these tCDE posets continue to hold in the piecewise-linear or birational setting? (Can still take limit of average along orbit even if order of rowmtion is not finite.)
- In a very recent paper *"Rowmotion in slow motion,"* Thomas and Williams define toggles for lattices beyond distributive lattices. Can we define tCDE posets beyond distributive lattices (maybe to prove certain intervals of weak order are CDE, as conjectured by RTY)?

Thank you!

References:

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