Refined Cyclic Sieving on Words and Tableaux

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based on joint work with
Connor Ahlbach and Brendon Rhoades

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Outline

- The cyclic sieving phenomenon (CSP) and refinements

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The Cyclic Sieving Phenomenon (CSP)

Definition (Reiner–Stanton–White, 2004)

Take $(X, C, f(q))$ where X is a finite set, C is a finite cyclic group acting on X, and $f(q) \in \mathbb{Z}_{\geq 0}[q]$.

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We say $(X, C, f(q))$ exhibits *the cyclic sieving phenomenon (CSP)* if for all $c \in C$ and roots of unity $\omega \in \mathbb{C}$ of the same order as c,

$$\#\{x \in X : c \cdot x = x\} = f(\omega).$$

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$$\#\{x \in X : c \cdot x = x\} = f(\omega).$$

(Equivalently, $f(\omega)$ is $\mathrm{Tr}_{\mathbb{C}\{X\}}(c)$. Note $f(1) = \#X$.)

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Example

Let $X = \binom{[n]}{k}$ and let $C = \mathbb{Z}/n$ act on X by addition mod n: if $n = 6, k = 3$, then

$$\overline{2} \cdot \{2, 3, 5\} = \{4, 5, 1\}.$$

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Recall:

- $\binom{n}{k}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}$
- $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$
- $[c]_q := 1 + q + \cdots + q^{c-1}$

CSP Refinements

Notation

Given $\mathrm{stat}\colon X \to \mathbb{Z}_{\geq 0}$, write

$$X^{\mathrm{stat}}(q) := \sum_{x \in X} q^{\mathrm{stat}(x)} \in \mathbb{Z}_{\geq 0}[q].$$

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Note $X^{\mathrm{stat}}(1) = \#X$.
In *many* CSP triples, $f(q) = X^{\mathrm{stat}}(q)$ for some stat.

Example

$\binom{n}{k}_q = \binom{[n]}{k}^{\mathrm{Sum}'}(q)$ where $\mathrm{Sum}'(A) = (\sum_{a \in A} a) - (1 + 2 + \cdots + k)$.

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Example

Take $X = \binom{[6]}{3}$, $Y = \mathbb{Z}/6 \cdot \{2, 3, 4\}$. Then $Y^{\text{Sum}'}(q) = 1 + 2q^3 + 2q^6 + q^9$, and

$$Y^{\text{Sum}'}(1) = 6, \qquad Y^{\text{Sum}'}(-1) = 0,$$
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We would need $Y^{\mathrm{Sum}'}(\omega_3) = 0$, not 6. So, $(Y, \mathbb{Z}/n, Y^{\mathrm{Sum}'}(q))$ does NOT quite refine the RSW CSP $(X, \mathbb{Z}/n, X^{\mathrm{Sum}'}(q))$.

A First Refinement Result

The *cyclic blocks* of a subset of $[n]$ are maximal sequences of adjacent elements in the subset, where 1 is considered adjacent to n. (Ex: $\{1, 2, 4, 6\} \subset [6]$ has two cyclic blocks, 612 and 4.)

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Here $X^{\mathrm{Sum}'}(q)$ is "equivalent" to $\binom{n}{k}_q$, so the unrefined triple is essentially RSW's.

Word Combinatorics

Definition

Given a word $w = w_1 \cdots w_n$ with letters $w_i \in \mathbb{Z}_{\geq 1}$, the *descent set* of w is

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(Ex: If $w = 323314$, then $\mathrm{Des}(w) = \{1, 4\}$, $\mathrm{maj}(w) = 1 + 4 = 5$, and $\alpha = (1, 1, 3, 1)$.)

A CSP on Words

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The triple $(W_\alpha, \mathbb{Z}/n, W_\alpha^{\text{maj}}(q))$ *exhibits the CSP.*

Remark

They actually proved a generalization valid for all finite Coxeter groups using Springer's regular elements, representation theory, coinvariant algebras, and len instead of maj.

Refined CSP on Words

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$$w^{(1)} = 1111 \qquad \mathrm{cdes}(w^{(1)}) = 0,$$
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We set $\mathrm{CDT}(143124114223) = (0, 2-0, 3-2, 5-3) = (0, 2, 1, 2)$.

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Notation

Let

$$W_{\alpha,\delta} := \text{words } w \text{ with content } \alpha \text{ and } \text{CDT}(w) = \delta.$$

Refined CSP on Words

Theorem (Ahlbach-S.)

$(W_{\alpha,\delta}, \mathbb{Z}/n, W^{\mathrm{maj}}_{\alpha,\delta}(q))$ refines the CSP triple $(W_{\alpha}, \mathbb{Z}/n, W^{\mathrm{maj}}_{\alpha}(q))$.

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Remark

Completely different proof than RSW. Combinatorial and largely recursive. Involves Carlitz-style decomposition, (more or less new) notion of "modular periodicity," a CSP extension lemma, a non-equivariant-but-fixed-point-preserving bijection, products of CSP's on sets and multisets.

Refined CSP on Words

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Theorem (Ahlbach-S.)

Let $\alpha \vDash n$ be a strong composition with m parts, $\delta \vDash k$, $n_i := |w^{(i)}|$, $k_i := \mathrm{cdes}(w^{(i)})$, $d := \gcd(n, k)$. Then, modulo $q^n - 1$,

$$W^{\mathrm{maj}}_{\alpha,\delta}(q) \equiv \frac{d}{\alpha_1}[n/d]_{q^d} \prod_{\ell=2}^{m} q^{k_\ell \alpha_\ell} \binom{n_{\ell-1} - k_{\ell-1}}{\delta_\ell}_q \left(\!\!\binom{k_\ell}{\alpha_\ell - \delta_\ell}\!\!\right)_{q^{-1}}$$

$$\equiv \frac{d}{\alpha_1}[n/d]_{q^d} q^{\eta} \prod_{\ell=2}^{m} \binom{n_{\ell-1} - k_{\ell-1}}{\delta_\ell}_q \left(\!\!\binom{k_\ell}{\alpha_\ell - \delta_\ell}\!\!\right)_{q}$$

where $\eta := \binom{k}{2} + \sum_{\ell=2}^{m} \binom{\delta_\ell}{2} - \alpha_1$.

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$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 & \\ \cline{1-2} 5 & 7 & \\ \cline{1-2}\end{array} \Rightarrow \mathrm{Des}(T) = \{2, 4, 6\}.$$

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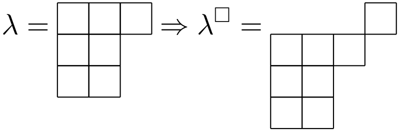
Definition

Given $\lambda \vdash n-1$, let $\lambda^{\square} \vdash n$ be the following "slightly skew partition":



Refined Sieving on Tableaux

Remark

Elizalde–Roichman (2017) defined a bijection
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(i) $\mathrm{cDes}(T) \cap [n-1] = \mathrm{Des}(T)$

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Proof reduces to showing $[n]_q$ divides $\mathrm{SYT}(\lambda^{\square})^{\mathrm{maj}}(q)$. Follows from

$$\begin{aligned}\mathrm{SYT}(\lambda^{\square})^{\mathrm{maj}}(q) &= \binom{n}{n-1,1}_q \mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)\,\mathrm{SYT}(\square)^{\mathrm{maj}}(q) \\ &= [n]_q\,\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q).\end{aligned}$$

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Remark

Showing $[n]_q \mid \mathrm{SYT}(\lambda^{\square}; k)^{\mathrm{maj}}(q)$ is significantly more involved. Uses an inner product formula of Adin–Reiner–Roichman (2017) for Elizalde–Roichman's cyclic descent extensions, a "change of basis," and the $W^{\mathrm{maj}}_{\alpha,\delta}(q)$ product formula above.

Further work

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- Give a representation-theoretic proof of $W_{\alpha,\delta}$ result

Thanks!

THANKS!