A minimaj-preserving crystal on ordered multiset partitions

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based on joint work with Georgia Benkart, Laura Colmenarejo, Pamela Harris, Rosa Orellana, Greta Panova, Martha Yip

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Project idea from talk by Brendon Rhoades in summer 2016 in Seoul

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  - crystal structure gives bijective proof

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$\pi = (157 \mid 24 \mid 56 \mid 468 \mid 13 \mid 123) \in \mathcal{OP}_{15,6}$

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Example

$\text{minimaj}(57.1 \mid 24 \mid 56. \mid 468. \mid 3.1 \mid 123) = 2 + 7 + 10 + 11 = 30$

Bijection with tuple of tableaux

Bijection:

$$\varphi\colon \mathcal{OP}_{n,k} \text{ with fixed } \ell \to \mathrm{SSYT}(1^{c_1}) \times \cdots \times \mathrm{SSYT}(1^{c_\ell}) \times \mathrm{SSYT}(\gamma)$$

descents in blocks

$$\pi \mapsto T_1 \times \cdots \times T_\ell \times T_{\ell+1}$$

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Example

$\pi = (124 \mid 45. \mid 3 \mid 46.1 \mid 23.1 \mid 1 \mid 25) \in \mathcal{OP}_{15,7}$ in minimaj order

$$\pi = (124 \mid 45. \mid 3 \mid 46.1 \mid 23.1 \mid 1 \mid 25) \mapsto \begin{array}{|c|}\hline 1 \\ \hline 5 \\ \hline\end{array} \times \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline\end{array} \times \begin{array}{|c|}\hline 6 \\ \hline\end{array} \times \begin{array}{cccc} & & 1 & 2 \\ & & 2 & \\ & 3 & 4 & \\ 1 & 4 & & \\ 2 & & & \\ 4 & & & \\ 5 & & & \end{array}$$

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Theorem (minimaj preserving crystal)

The operators $\tilde{e}_i, \tilde{f}_i$, and wt impose an $\mathfrak{sl}_r$-crystal structure on $\mathcal{OP}^{(r)}_{n,k}$. In addition, $\tilde{e}_i$ and $\tilde{f}_i$ preserve the minimaj statistic.

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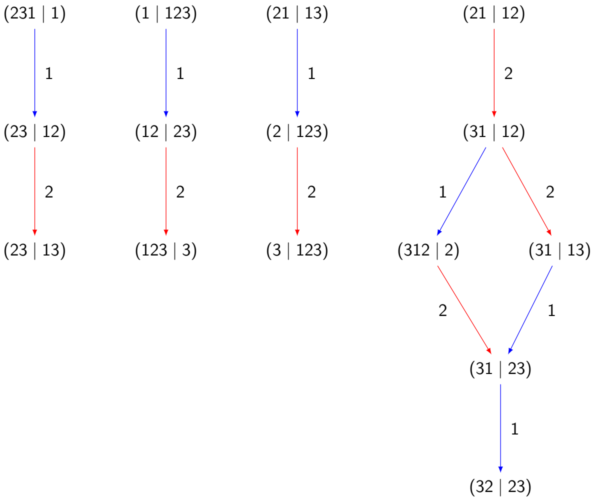
Corollary (Schur expansion)

$$\mathrm{Val}_{n,k-1}(\mathbf{x};0,t) = \sum_{\substack{\pi \in \mathcal{OP}_{n,k} \\ \tilde{e}_i(\pi)=0 \ \forall\, i}} t^{\mathrm{minimaj}(\pi)} s_{\mathrm{wt}(\pi)}$$

Crystal structure on ordered multiset partitions



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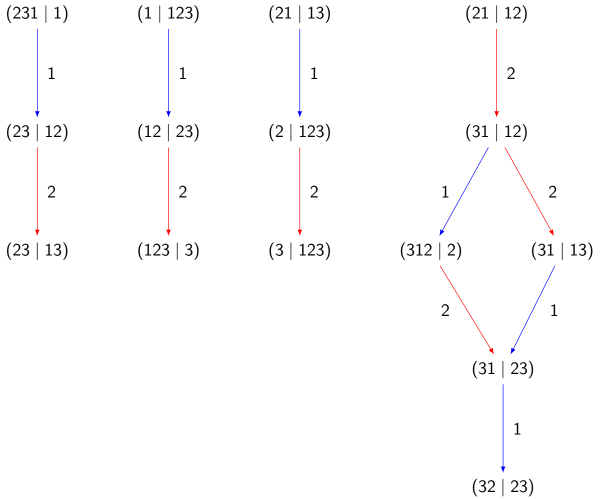

minimaj $2, 0, 1, 1$

$$\mathrm{Val}_{4,1}(\mathbf{x}; 0, t) = (1 + t + t^2)\, s_{(2,1,1)}(\mathbf{x}) + t\, s_{(2,2)}(\mathbf{x})$$

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$$w = 751 \mid 42 \mid 65 \mid 864 \mid 31 \mid 321$$
$$v = 001 \mid 12 \mid 23 \mid 334 \mid 45 \mid 556$$

so that $\mathrm{maj}(\pi) = 0 + 0 + 1 + 2 + 3 + 3 + 4 + 4 + 5 + 5 = 27$

Bijection

Theorem

The map $\psi\colon \mathcal{OP}_{n,k} \to \mathcal{OP}_{n,k}$ *defined by*

$$\psi(\pi) = \mathsf{L}(\mathsf{read}(\varphi(\pi))) \qquad \text{for } \pi \in \mathcal{OP}_{n,k} \text{ in minimaj order}$$

is a bijection and

$$\mathsf{minimaj}(\pi) = \mathsf{maj}(\psi(\pi))$$

φ bijection from $\mathcal{OP}_{n,k}$ to tuple of (skew) tableaux

read column reading word

L left shift map

read

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$\pi = (1 \mid 56. \mid 4. \mid 37.12 \mid 2.1 \mid 1 \mid 34) \in \mathcal{OP}_{13,7}$ in minimaj order

$$T^\bullet = \varphi(\pi) = \begin{array}{c} 1 \\ 4 \end{array} \times \begin{array}{c} 1 \\ 2 \end{array} \times \begin{array}{c} 7 \end{array} \times \emptyset \times \begin{array}{ccc} & 1 & 3 \\ & 2 & \\ & 3 & \\ & 4 & \\ 1 & 5 & \\ 6 & & \end{array}$$

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Lemma

read *is invertible.*

Left shift

$\pi' = \text{read}(\varphi(\pi))$

Suppose π' has blocks $\pi'_{p_m}, \ldots, \pi'_{p_1}$ with $1 \leqslant p_m < \cdots < p_2 < p_1 < k$:

- π_{p_i} empty or
- π_{p_i} has descent at the end

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Left shift operation L on π'

1. Set $\mathrm{L}^{(0)}(\pi') = \pi'$.

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   - S_i = sequence of elements starting immediately to right of block π'_{p_i} in $\mathrm{L}^{(i-1)}(\pi')$ up to and including the p_i-th descent after the block π'_{p_i}

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2. Suppose $\mathrm{L}^{(i-1)}(\pi')$ for $1 \leqslant i \leqslant m$ is defined:
   - By induction, the p_i-th block of $\mathrm{L}^{(i-1)}(\pi')$ is π'_{p_i}.
   - S_i = sequence of elements starting immediately to right of block π'_{p_i} in $\mathrm{L}^{(i-1)}(\pi')$ up to and including the p_i-th descent after the block π'_{p_i}
   - $\mathrm{L}^{(i)}(\pi')$ = move each element in S_i in $\mathrm{L}^{(i-1)}(\pi')$ one block left

Left shift

$\pi' = \text{read}(\varphi(\pi))$

Suppose π' has blocks $\pi'_{p_m}, \ldots, \pi'_{p_1}$ with $1 \leqslant p_m < \cdots < p_2 < p_1 < k$:

- π_{p_i} empty or
- π_{p_i} has descent at the end

Definition

Left shift operation L on π'

1. Set $\mathrm{L}^{(0)}(\pi') = \pi'$.
2. Suppose $\mathrm{L}^{(i-1)}(\pi')$ for $1 \leqslant i \leqslant m$ is defined:
   - By induction, the p_i-th block of $\mathrm{L}^{(i-1)}(\pi')$ is π'_{p_i}.
   - S_i = sequence of elements starting immediately to right of block π'_{p_i} in $\mathrm{L}^{(i-1)}(\pi')$ up to and including the p_i-th descent after the block π'_{p_i}
   - $\mathrm{L}^{(i)}(\pi')$ = move each element in S_i in $\mathrm{L}^{(i-1)}(\pi')$ one block left

$\mathrm{L}(\pi') := \mathrm{L}^{(m)}(\pi')$

Left shift

Example

$$\pi' = (4.1 \mid 2.1 \mid 7. \mid \emptyset \mid 6.1 \mid 5.4.3.2.1 \mid 3) \quad p_1 = 4, S_1 = 61543$$

Left shift

Example

$$\pi' = (4.1 \mid 2.1 \mid 7. \mid \emptyset \mid 6.1 \mid 5.4.3.2.1 \mid 3) \quad p_1 = 4, S_1 = 61543$$

$$\mathsf{L}^{(1)}(\pi') = (4.1 \mid 2.1 \mid 7. \mid 6.1 \mid 5.4.3. \mid 2.1 \mid 3) \quad p_2 = 3, S_2 = 6154$$

Left shift

Example

$$\pi' = (4.1 \mid 2.1 \mid 7. \mid \emptyset \mid 6.1 \mid 5.4.3.2.1 \mid 3) \quad p_1 = 4, S_1 = 61543$$

$$\mathrm{L}^{(1)}(\pi') = (4.1 \mid 2.1 \mid 7. \mid 6.1 \mid 5.4.3. \mid 2.1 \mid 3) \quad p_2 = 3, S_2 = 6154$$

$$\mathrm{L}^{(2)}(\pi') = (4.1 \mid 2.1 \mid 7.6.1 \mid 5.4. \mid 3. \mid 2.1 \mid 3)$$

Left shift

Example

$$\pi' = (4.1 \mid 2.1 \mid 7. \mid \emptyset \mid 6.1 \mid 5.4.3.2.1 \mid 3) \quad p_1 = 4, S_1 = 61543$$

$$\mathrm{L}^{(1)}(\pi') = (4.1 \mid 2.1 \mid 7. \mid 6.1 \mid 5.4.3. \mid 2.1 \mid 3) \quad p_2 = 3, S_2 = 6154$$

$$\mathrm{L}^{(2)}(\pi') = (4.1 \mid 2.1 \mid 7.6.1 \mid 5.4. \mid 3. \mid 2.1 \mid 3)$$

$$\mathrm{maj}(\mathrm{L}^{(i)}(\pi')) = \begin{cases} \mathrm{maj}(\mathrm{L}^{(i-1)}(\pi')) - p_i + 1, & \text{if } \pi'_{p_i} = \emptyset \\ \mathrm{maj}(\mathrm{L}^{(i-1)}(\pi')) - p_i, & \text{if } \pi'_{p_i} \text{ descent at end of block} \end{cases}$$

Left shift

Example

$$\pi' = (4.1 \mid 2.1 \mid 7. \mid \emptyset \mid 6.1 \mid 5.4.3.2.1 \mid 3) \quad p_1 = 4, S_1 = 61543$$
$$\mathrm{L}^{(1)}(\pi') = (4.1 \mid 2.1 \mid 7. \mid 6.1 \mid 5.4.3. \mid 2.1 \mid 3) \quad p_2 = 3, S_2 = 6154$$
$$\mathrm{L}^{(2)}(\pi') = (4.1 \mid 2.1 \mid 7.6.1 \mid 5.4. \mid 3. \mid 2.1 \mid 3)$$

$$\mathrm{maj}(\mathrm{L}^{(i)}(\pi')) = \begin{cases} \mathrm{maj}(\mathrm{L}^{(i-1)}(\pi')) - p_i + 1, & \text{if } \pi'_{p_i} = \emptyset \\ \mathrm{maj}(\mathrm{L}^{(i-1)}(\pi')) - p_i, & \text{if } \pi'_{p_i} \text{ descent at end of block} \end{cases}$$

Example

$\mathrm{maj}(\pi') = 28$, $\mathrm{maj}(\mathrm{L}^{(1)}(\pi')) = 25$, $\mathrm{maj}(\mathrm{L}(\pi')) = 22 = \mathrm{minimaj}(\pi)$

Thank you !