Orbits of plane partitions of exceptional Lie type

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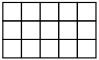
Based on joint work with Holly Mandel (Berkeley)
arXiv:1712.09180

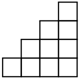
Minuscule posets

- The **minuscule** posets are the following 5 families:



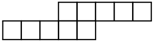
Rectangle

type A



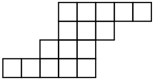
Shifted staircase

type $B/C/D$



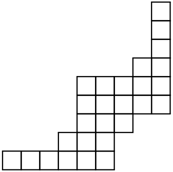
Propeller

type D



Cayley-Moufang

type E_6

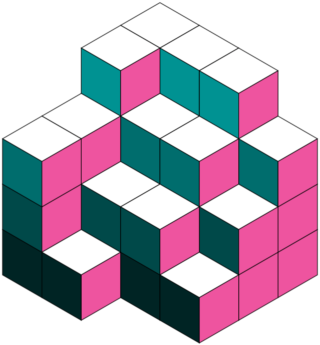


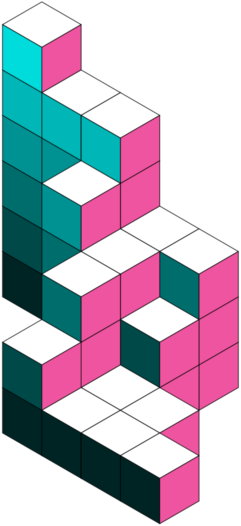
Freudenthal

type E_7

- Minuscule posets describe the Schubert cell decompositions of certain generalized Grassmannians, as well as certain representations of Lie groups

We study *plane partitions* over these posets





Counting plane partitions

- How many plane partitions of height at most k?
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$$f_{\mathcal{P}}^k(q) = \sum_{\mathcal{I} \in \mathrm{PP}^k(\mathcal{P})} q^{|\mathcal{I}|}$$

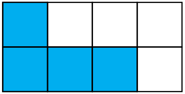
- For $\mathcal{P}$ minuscule, $f_{\mathcal{P}}^k$ has a beautiful product formula (Proctor '84):

$$f_P^k(q) = \prod_{\mathrm{x} \in \mathcal{P}} \frac{(1 - q^{\mathrm{rk}(\mathrm{x})+k})}{(1 - q^{\mathrm{rk}(\mathrm{x})})},$$

  where $\mathrm{rk}(\mathrm{x})$ denotes the size of the largest chain in $\mathcal{P}$ with maximum element x.

Rowmotion of partitions

- Fix an $a \times b$ rectangle
- Consider ways to stack 1×1 boxes in the lower left corner

$\lambda =$ 

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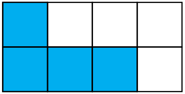
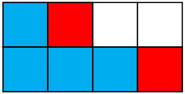
$\lambda =$ 

- Look at all places where you could add a single box



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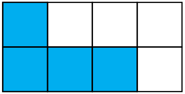
$\lambda =$ 

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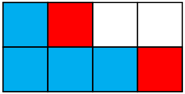
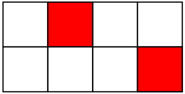


- Remove old boxes



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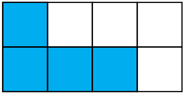
$\lambda =$ 

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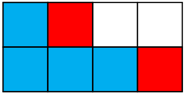

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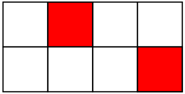
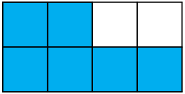
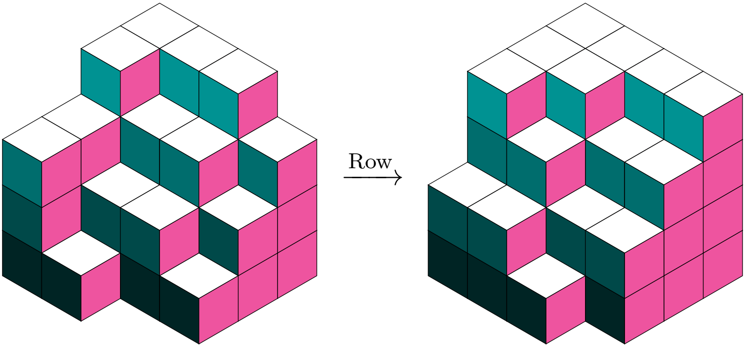


- Add just enough boxes to support the remaining boxes

$\mathrm{Row}(\lambda) =$ 

Rowmotion of plane partitions



Cyclic sieving

- Evaluating $f^k_{\mathcal{P}}(q)$ at roots-of-unity gives additional enumerations!
- $f^k_{\mathcal{P}}(1) = \#\mathrm{PP}^k(\mathcal{P})$

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Theorem (Rush–Shi '13)

Let n be the period of Row on $\mathrm{PP}^k(\mathcal{P})$ and let ζ be a primitive nth root-of-unity. For $\mathcal{P}$ minuscule and $\mathbf{k} \leq \mathbf{2}$,

$$f^k_{\mathcal{P}}(\zeta^d) = \#\mathrm{PP}^k(\mathcal{P})^{\mathrm{Row}^d}.$$

- The theorem does not extend to $k > 2$ in general.

Cyclic sieving at greater heights

Theorem (Rush–Shi '11, unpub.)

Let n be the period of Row *on* $\mathsf{PP}^k(\mathcal{P})$ *and let* ζ *be a primitive nth root-of-unity. For* $\mathcal{P}$ *a* **propeller** *and* **all** k,

$$f^k_{\mathcal{P}}(\zeta^d) = \#\mathsf{PP}^k(\mathcal{P})^{\mathrm{Row}^d}.$$

Conjecture (Rush–Shi '13)

Let n be the period of Row *on* $\mathsf{PP}^k(\mathcal{P})$ *and let* ζ *be a primitive nth root-of-unity. For* $\mathcal{P}$ *the* **Cayley-Moufang or Freudenthal poset** *and* **all** k,

$$f^k_{\mathcal{P}}(\zeta^d) = \#\mathsf{PP}^k(\mathcal{P})^{\mathrm{Row}^d}.$$

Main results

Theorem (Mandel–P '17)

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Key reformulation

Theorem (Dilks-P-Striker '17, Dilks-Striker-Vorland '17)

For $\mathcal{P}$ minuscule, there is an equivariant bijection

$$\mathsf{PP}^k(\mathcal{P}) \Longleftrightarrow \mathrm{Inc}^m(\mathcal{P})$$

with Row acting on $\mathsf{PP}^k(\mathcal{P})$ and Pro acting on $\mathrm{Inc}^m(\mathcal{P})$.

- The right-side is combinatorics extracted from K-theoretic Schubert calculus (Thomas-Yong '09, ...).
- It is easier for us to understand!

Promotion of increasing tableaux

$$
\begin{array}{c}
< \\
\vee \ \begin{array}{|c|c|c|} \hline 6 & 7 & 8 \\ \hline 2 & 4 & 7 \\ \hline 1 & 2 & 3 \\ \hline \end{array} \ \in \mathrm{Inc}^8(3 \times 3)
\end{array}
$$

Promotion of increasing tableaux

6	7	8
2	4	7
$\bullet$	2	3

Promotion of increasing tableaux

6	7	8
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Promotion of increasing tableaux

6	7	8
•	4	7
2	3	•

Promotion of increasing tableaux

6	7	8
●	4	7
2	3	●

Promotion of increasing tableaux

6	7	8
4	•	7
2	3	•

Promotion of increasing tableaux

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4	7	•
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6	8	•
4	7	8
2	3	7

Promotion of increasing tableaux

6	8	•
4	7	8
2	3	7

Promotion of increasing tableaux

6	8	9
4	7	8
2	3	7

Promotion of increasing tableaux

5	7	8
3	6	7
1	2	6

$\in \mathrm{Inc}^8(3 \times 3)$

Deflation

6	7	8
2	4	7
1	2	3

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→ Deflation →

5	6	7
2	4	6
1	2	3

$\in \mathrm{Inc}^7_{\mathrm{gl}}(3 \times 3)$

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Proposition (Mandel-P '17)

- *If $1 \in T$, then promotion commutes with deflation.*
- *If $1 \notin T$, then promotion is given by decrementing each entry.*

Controlling promotion

Theorem (Mandel-P '17)

Let $T \in \mathrm{Inc}^m(\mathcal{P})$*.*

Let τ *be the promotion period of* $\mathrm{Deflation}(T) \in \mathrm{Inc}_{\mathrm{gl}}^{m'}(\mathcal{P})$ *and let* ℓ *be the cyclic-rotation period of* $\mathrm{Content}(T)$*.*

Then, the promotion period of T *is*

$$\frac{\ell \tau}{\gcd(\ell m'/m, \tau)}.$$

- Thus, it suffices to understand promotion as restricted to gapless tableaux.
- But there are only finitely-many such for any fixed $\mathcal{P}$!

Finishing the proof

- We find there are:

  3 gapless tableaux for any propeller,

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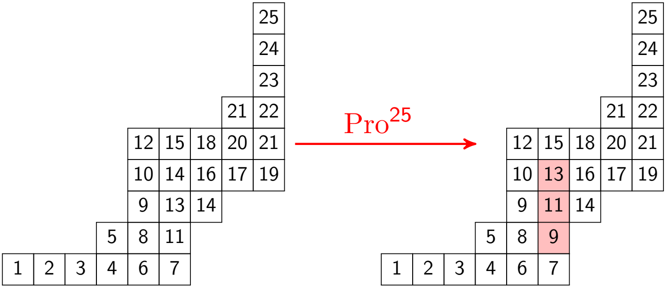
- We compute the promotion periods of all of these.
- Finally, what remains is essentially arithmetic with q-integers...
- This proves CSPs for propellers and for the Cayley-Moufang poset.

What's wrong with the Freudenthal poset?

- Why does the conjectured CSP fail for the Freudenthal poset?
- It seems even the period of promotion/rowmotion is not the predicted one!



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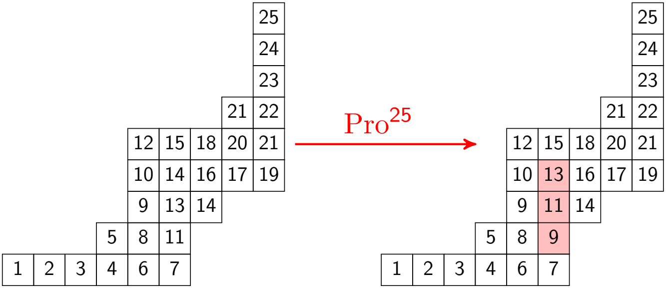

- Order in this case is 75.
- But plugging in 75th roots-of-unity into the appropriate *q*-enumerator doesn't even yield integers.

Thanks!

Thank you!!