Dynamical algebraic combinatorics: Resonance

Jessica Striker

North Dakota State University

March 14, 2018

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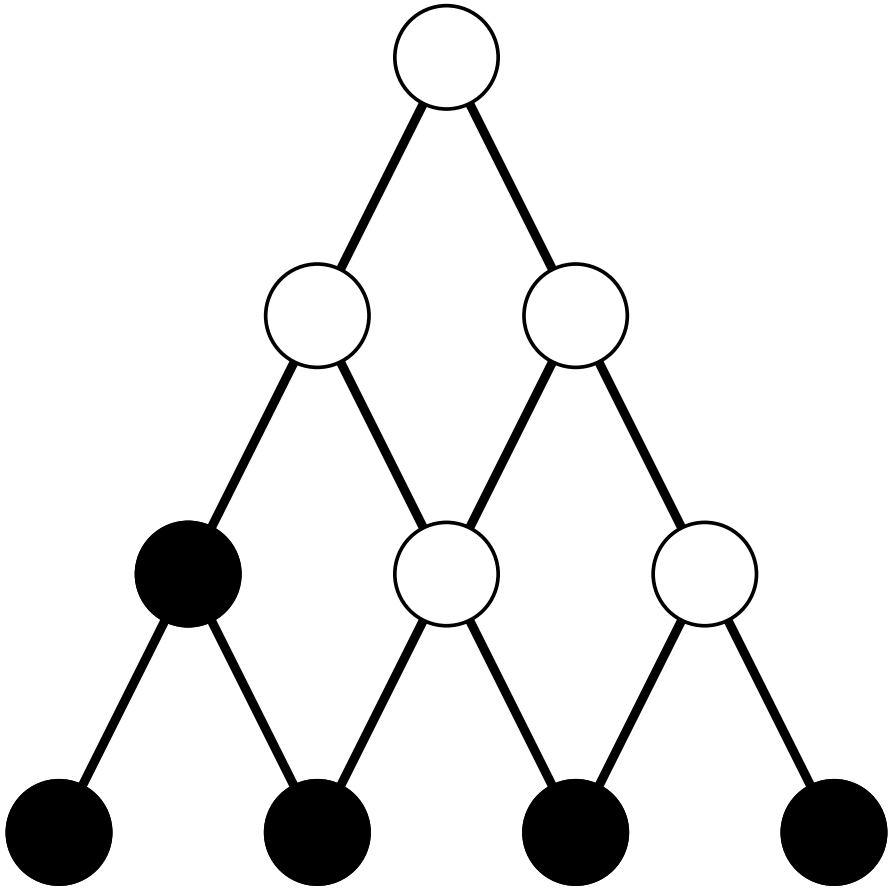
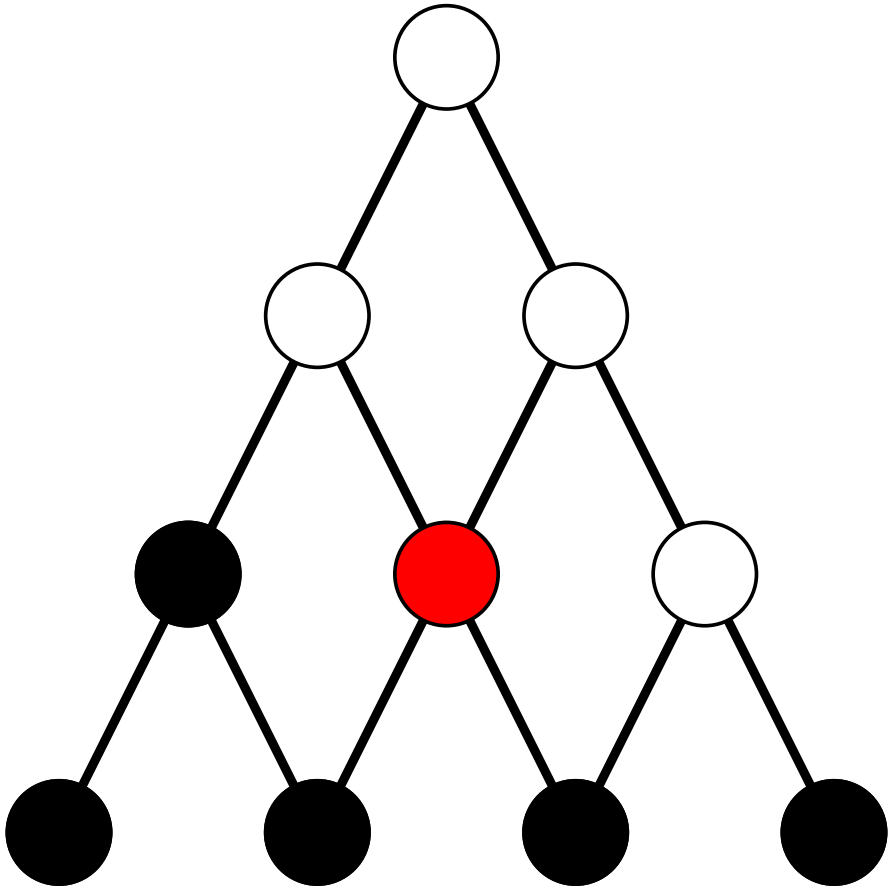
Toggles act on order ideals

Define a toggle, t_e, for each $e \in P$.



Toggles act on order ideals

Toggles t_e add e when possible.



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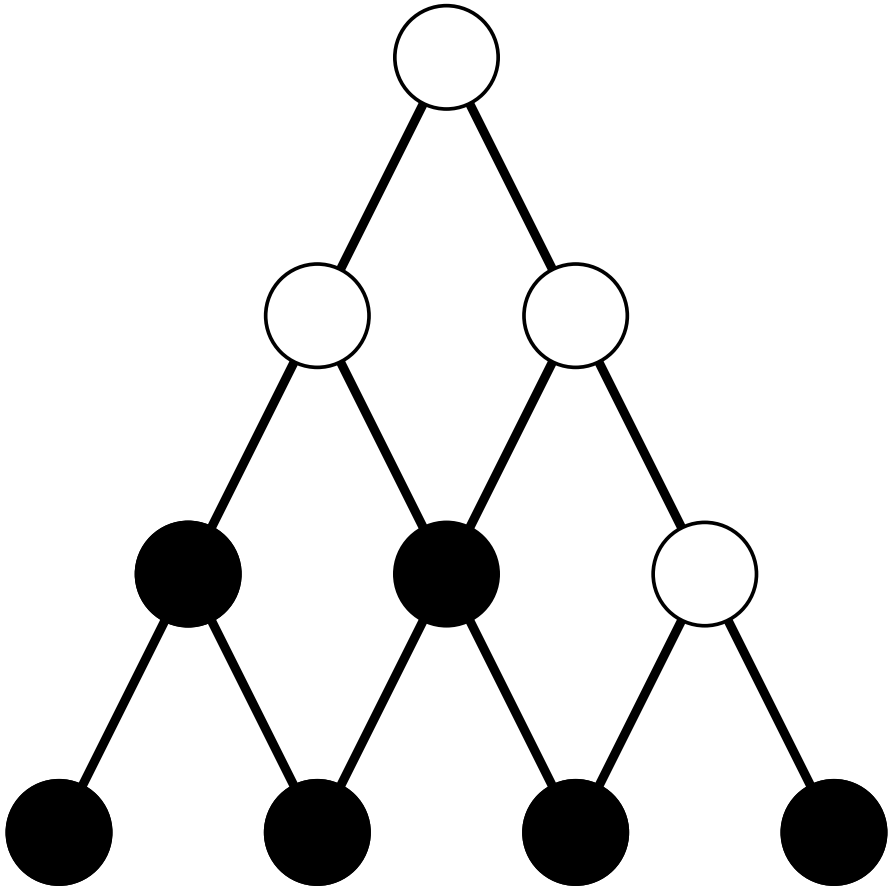

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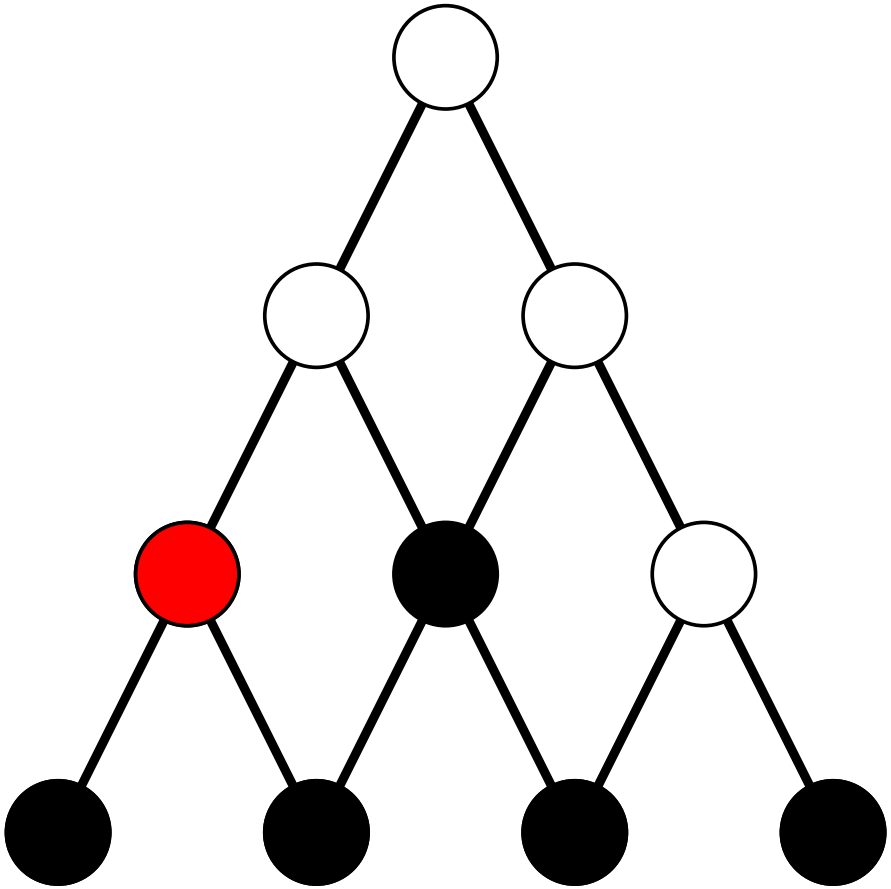

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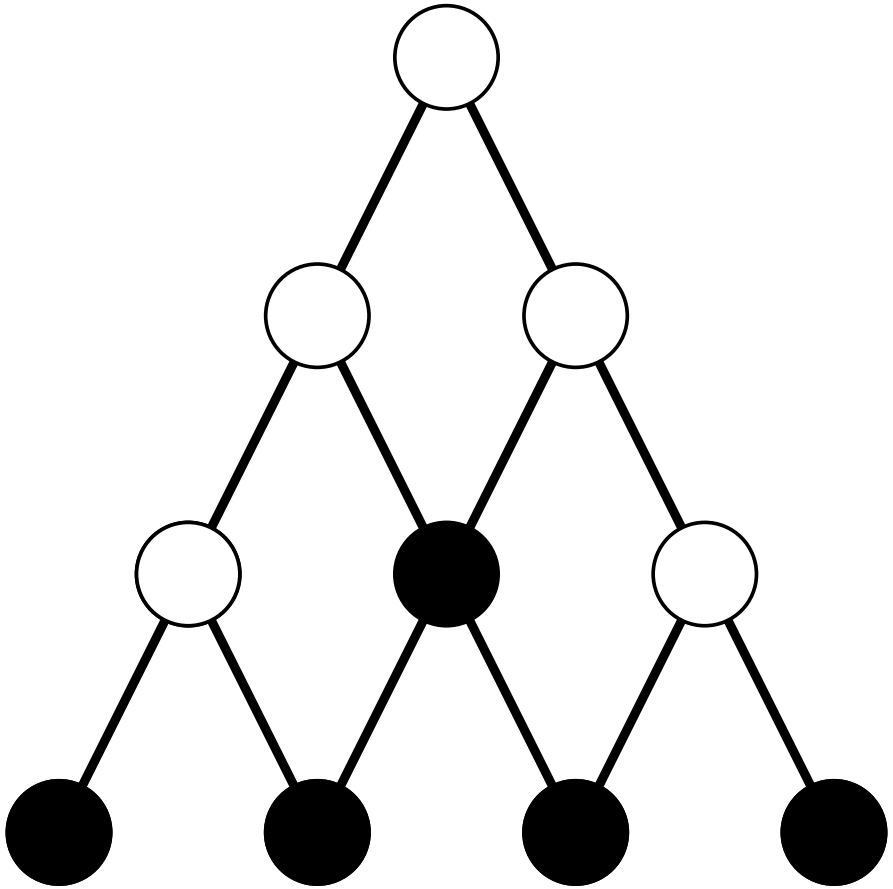

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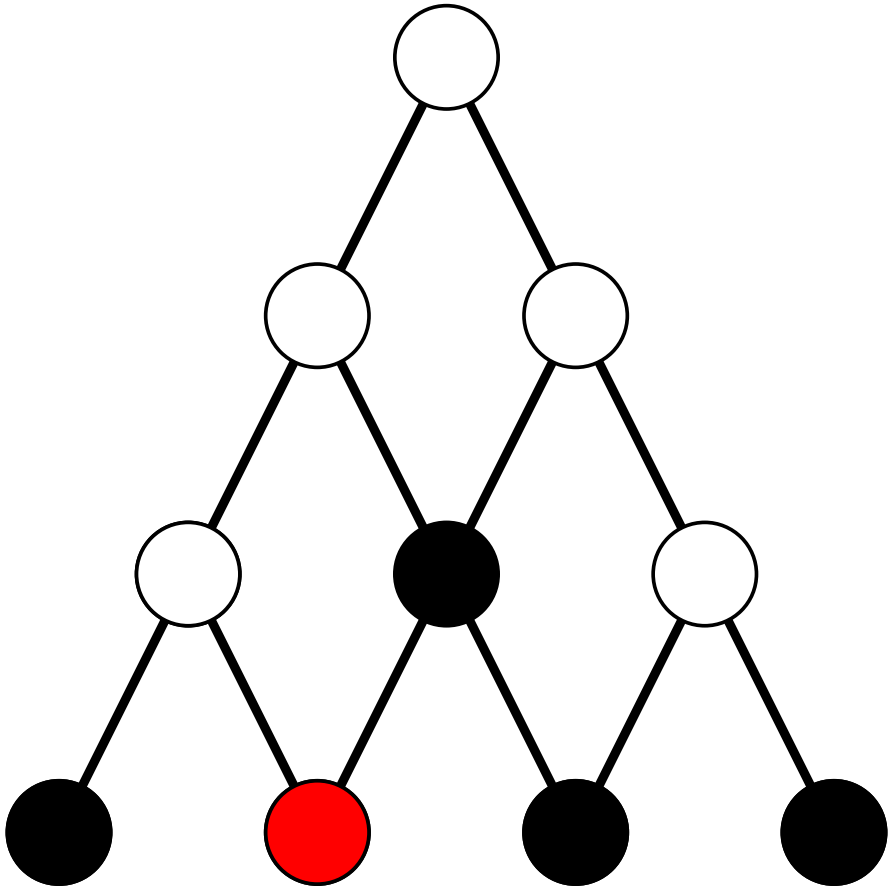

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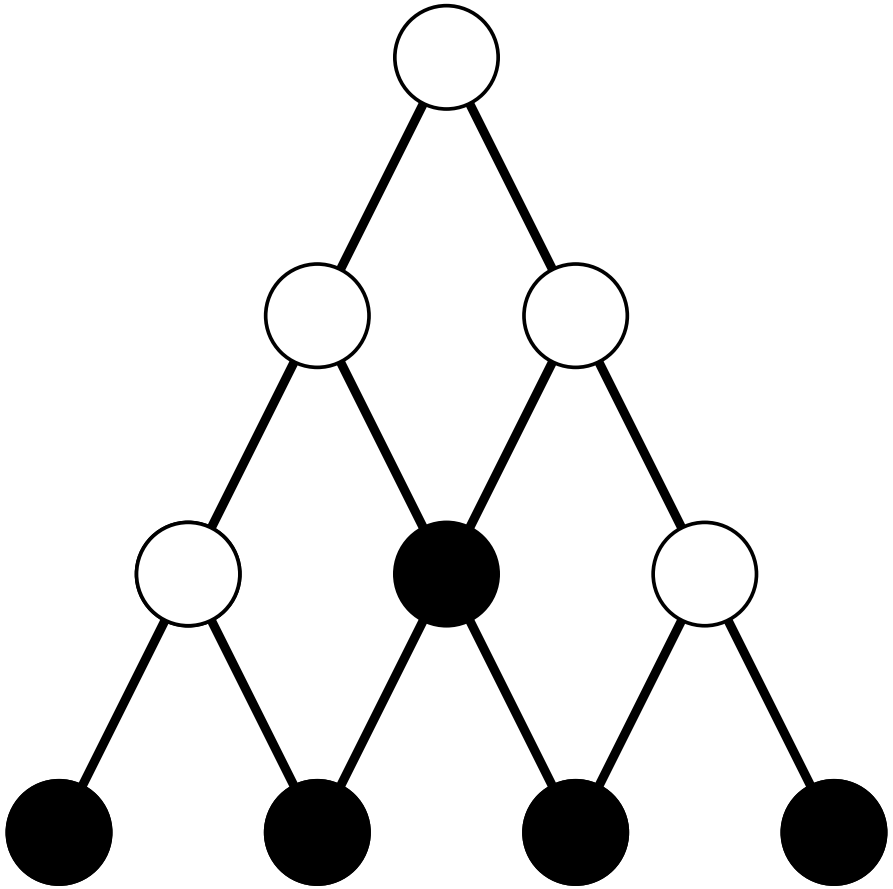
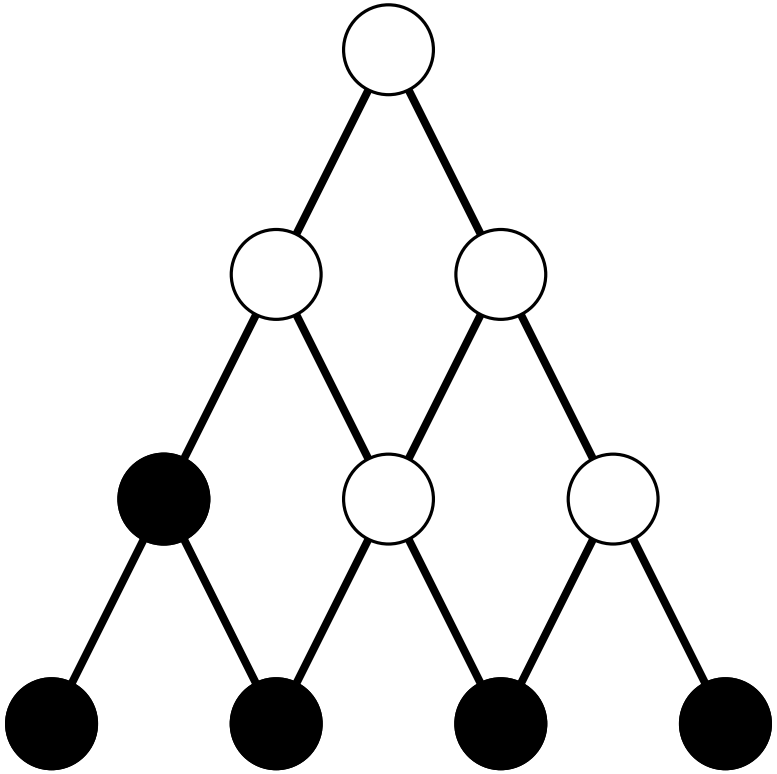


Rowmotion

Definition

The **rowmotion** of an order ideal X is the order ideal generated by the minimal elements of P not in X.

An order ideal X



Rowmotion

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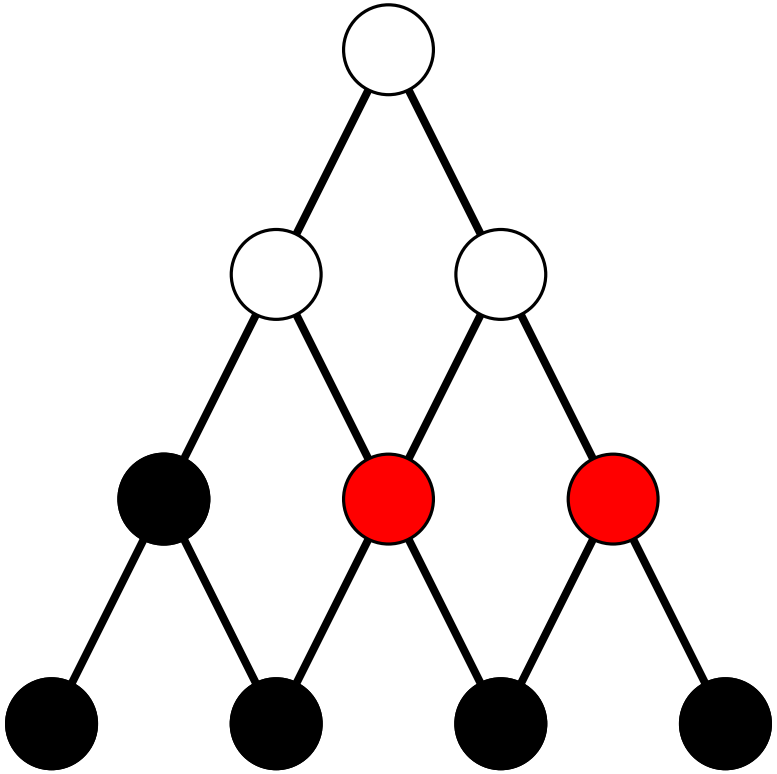
The **rowmotion** of an order ideal X is the order ideal generated by the minimal elements of P not in X.

Find the **minimal** elements of P not in X.

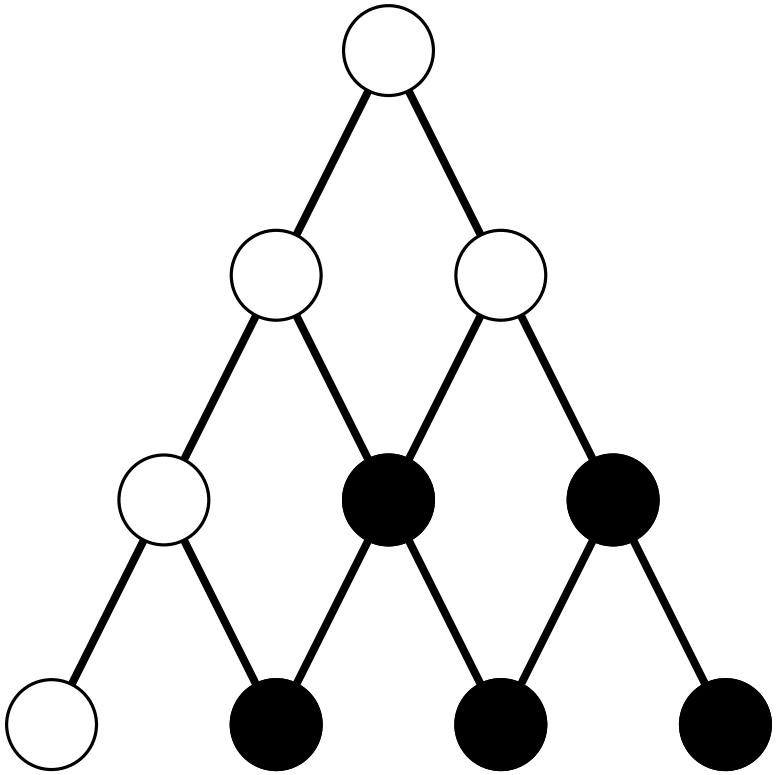


Rowmotion

Definition

The **rowmotion** of an order ideal X is the order ideal generated by the minimal elements of P not in X.

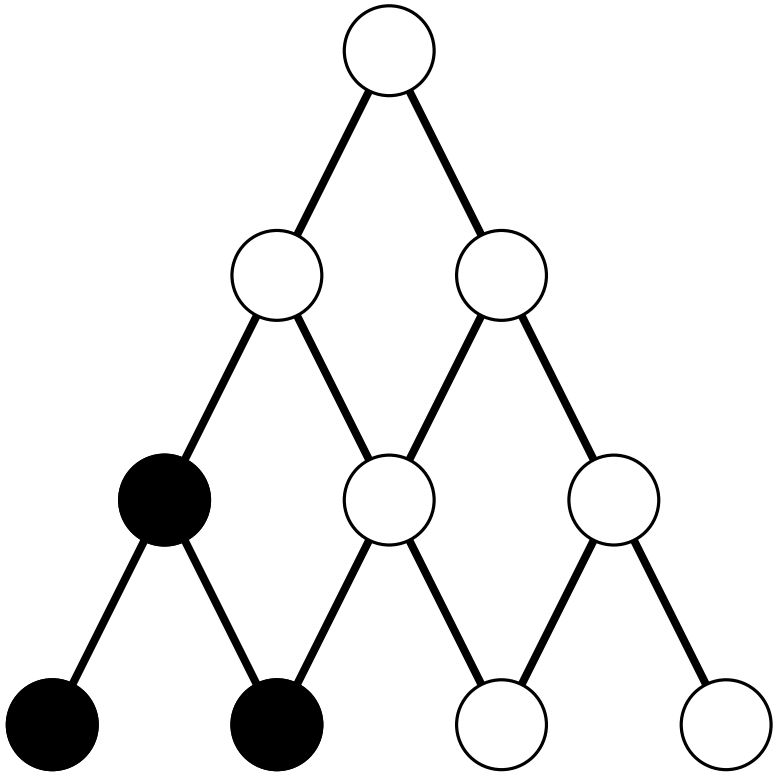
Use them to generate a new order ideal **Row**(**X**).



Rowmotion on order ideals is a product of toggles

Theorem (P. Cameron and D. Fon-der-Flaass 1995)

Let P be a poset and X an order ideal of P. Then rowmotion on X is the order ideal obtained from X by toggling the elements of P from top to bottom.



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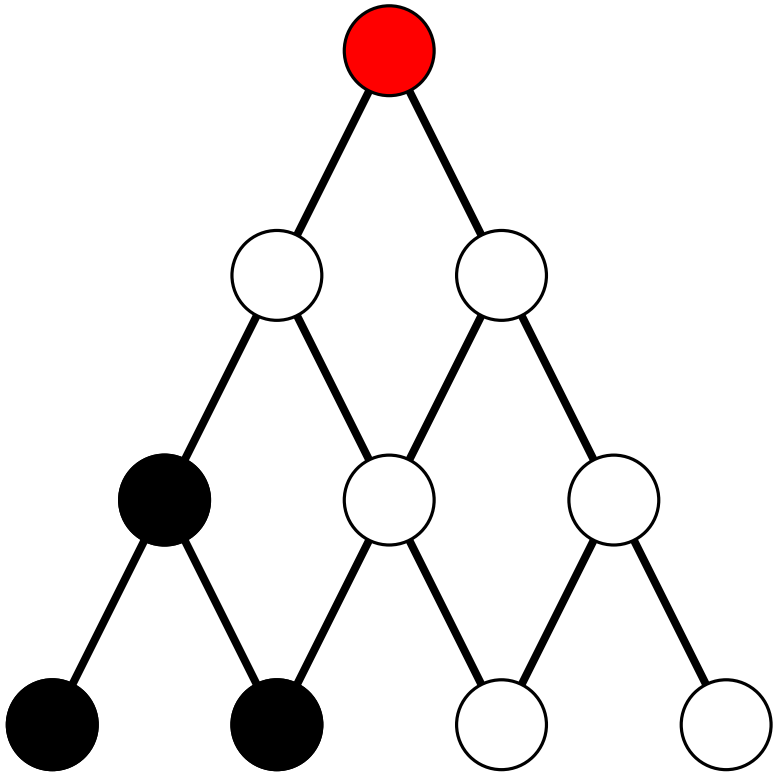

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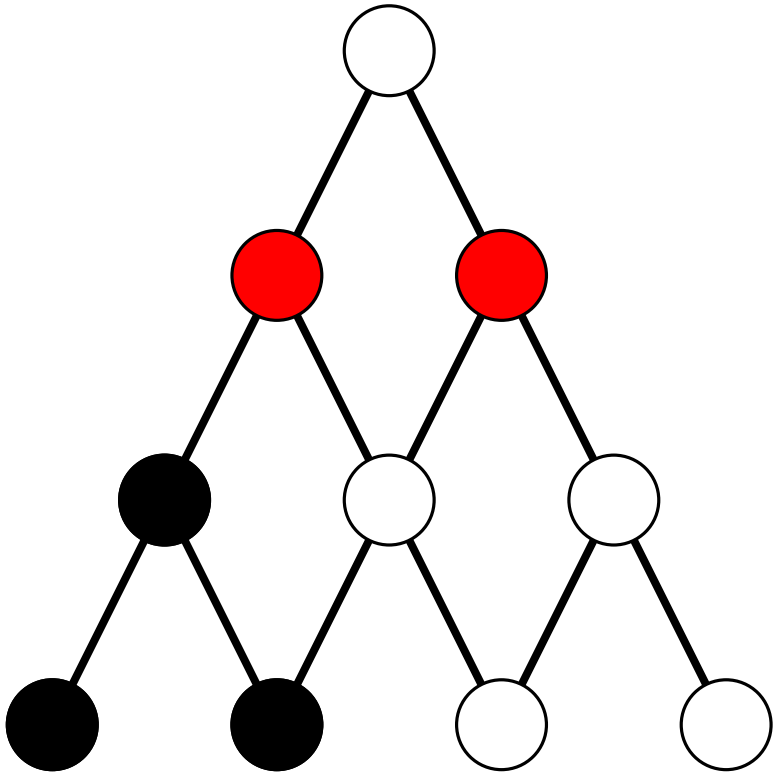

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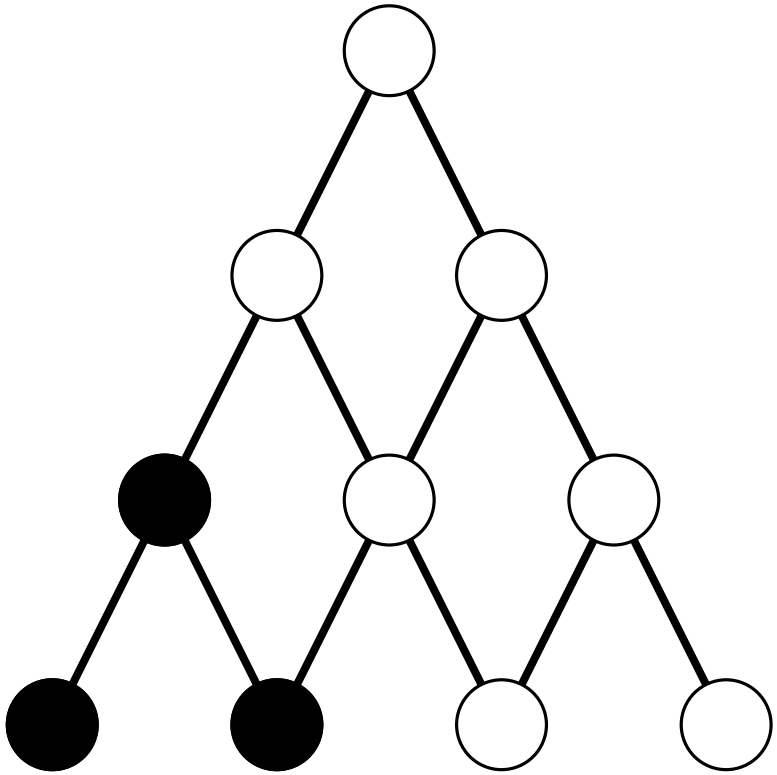

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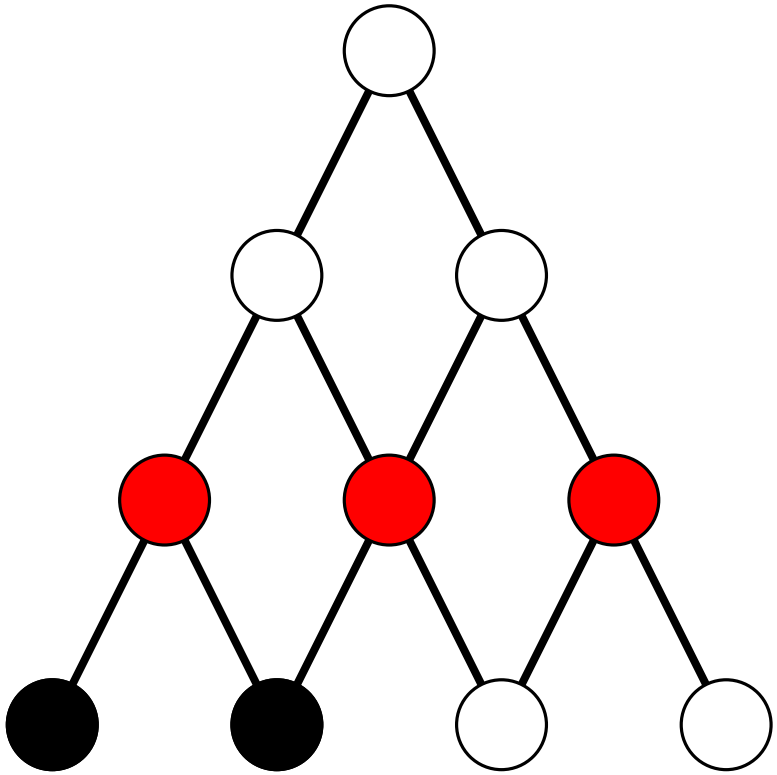

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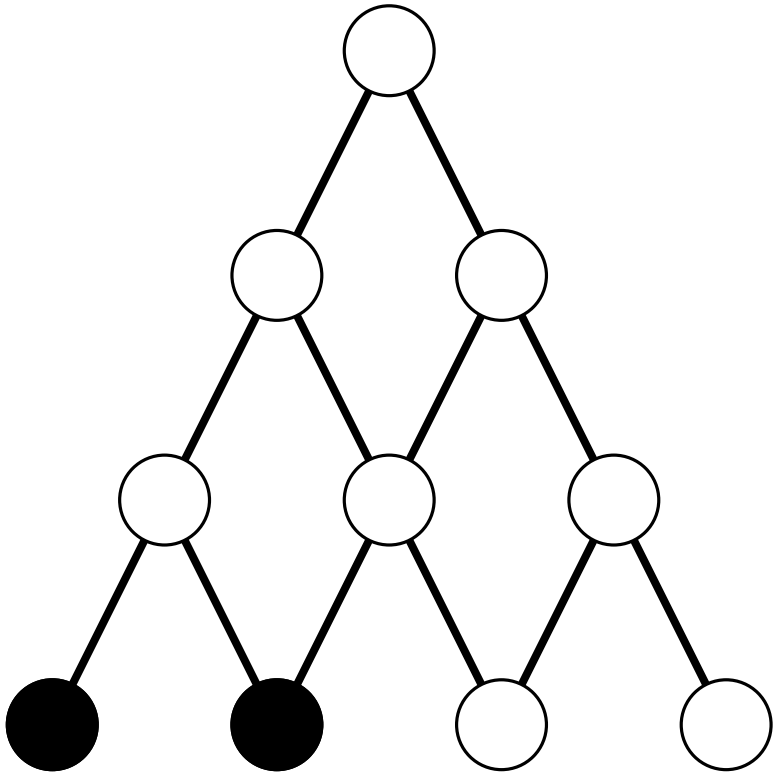

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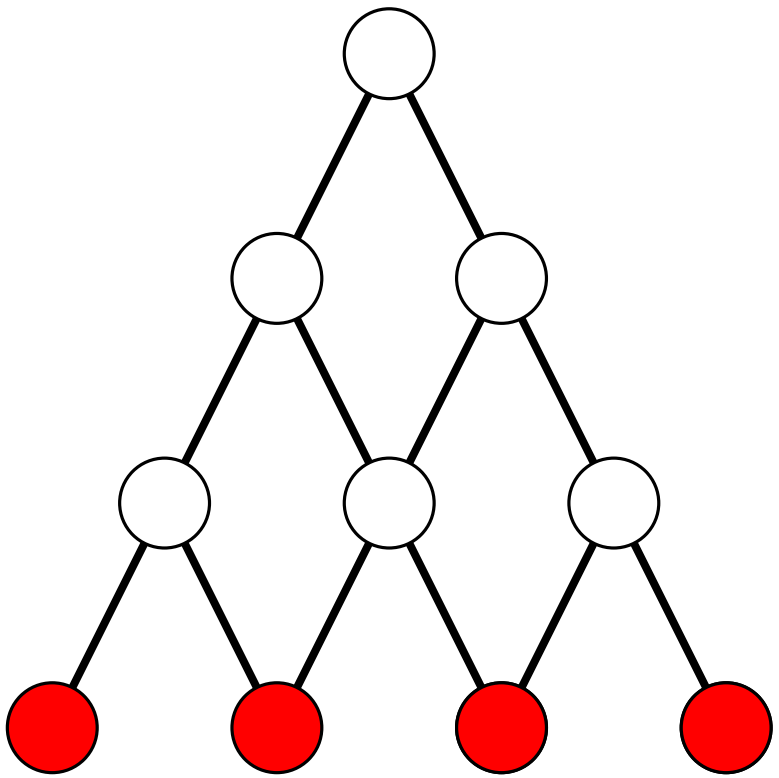

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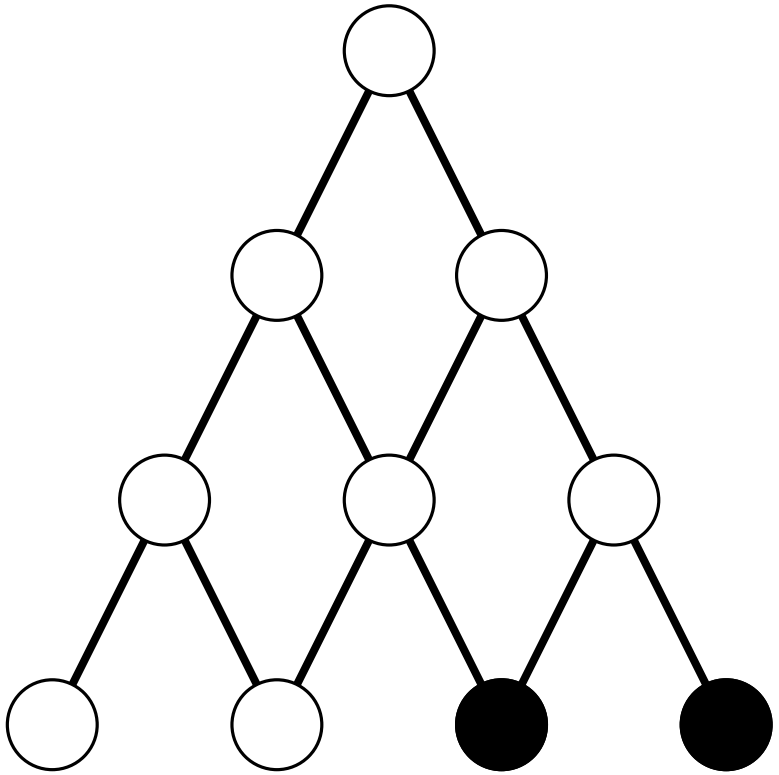


Promotion and rowmotion are conjugate actions

Theorem (N. Williams and S. 2012)

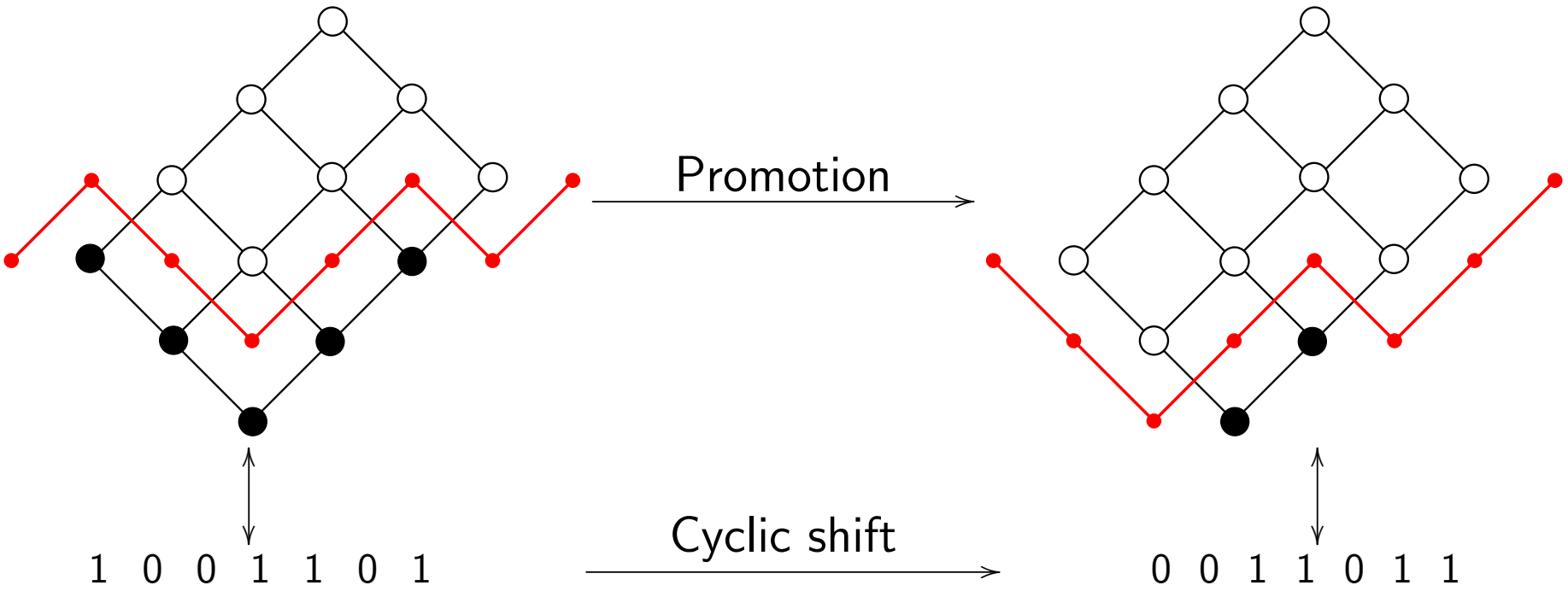
In any ranked poset, there are equivariant bijections between the order ideals under rowmotion (toggle top to bottom) and promotion (toggle left to right).

In an *equivariant* bijection, the orbit structure is preserved.

Rowmotion in rectangular posets $\mathbf{a} \times \mathbf{b}$

Corollary (N. Willilams and S. 2012)

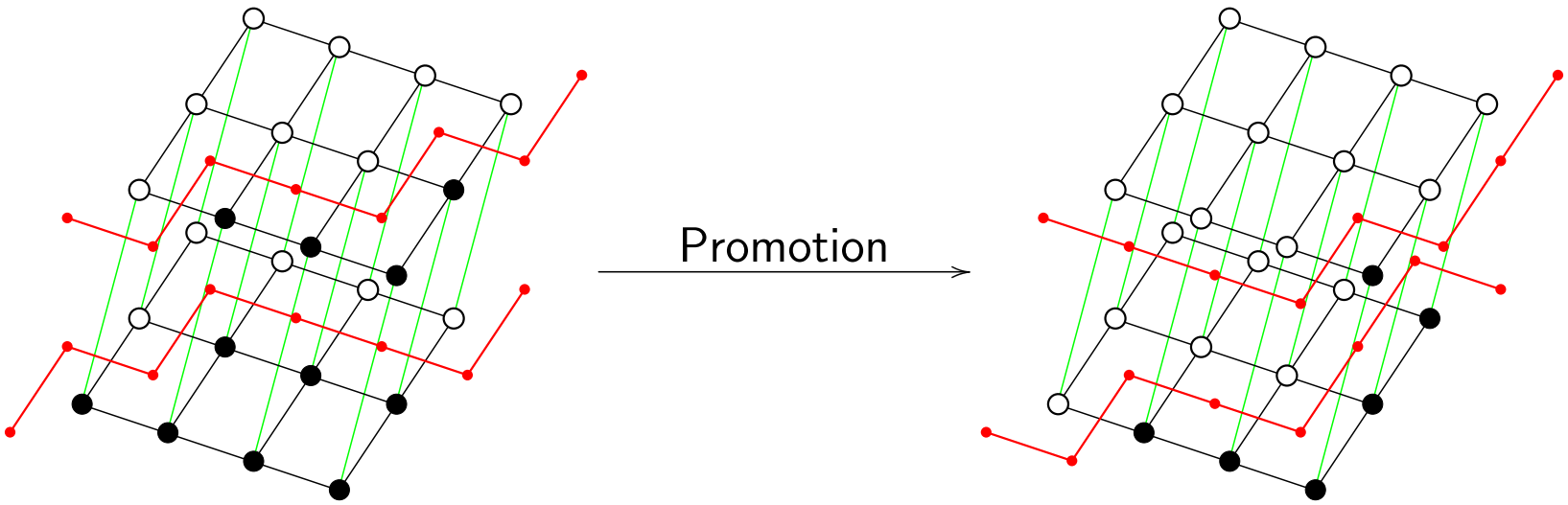
There is an equivariant bijection between order ideals of $\mathbf{a} \times \mathbf{b}$ *under rowmotion and binary words of length* $a + b$ *with* b *ones under a cyclic shift. So rowmotion has order* $a + b$ *and exhibits the cyclic sieving phenomenon.*



Rowmotion in $\mathbf{a} \times \mathbf{b} \times \mathbf{2}$

Theorem (N. Williams and S. 2012)

There is an equivariant bijection between $J(\mathbf{a} \times \mathbf{b} \times \mathbf{2})$ under rowmotion and noncrossing partitions of $a + b + 1$ into $b + 1$ blocks under rotation. So rowmotion has order $a + b + 1$ and exhibits the cyclic sieving phenomenon.



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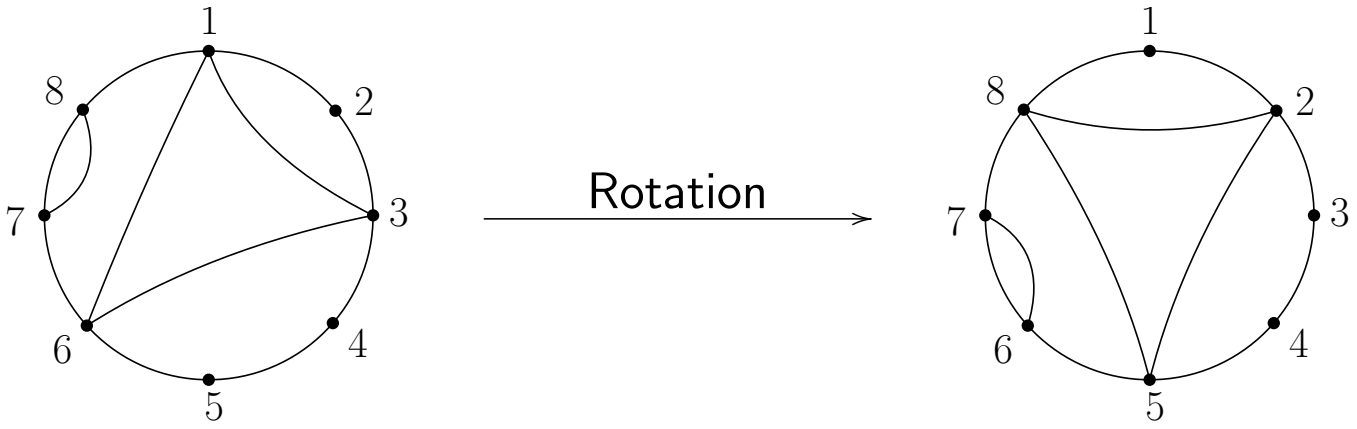

What is the order of rowmotion on $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$?

- The order of rowmotion on $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{b} \times \mathbf{1}$ is $a + b$.
- The order of rowmotion on $\mathbf{a} \times \mathbf{b} \times \mathbf{2}$ is $a + b + 1$.
- The order of rowmotion on $\mathbf{a} \times \mathbf{b} \times \mathbf{3}$ is $a + b + 2$ for as high as we can check.
- This may lead one to conjecture that the order of rowmotion on $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$ is $a + b + c - 1$.
- But the orbits of rowmotion on $\mathbf{4} \times \mathbf{4} \times \mathbf{4}$ are of size 11 and 33.

Cyclic sieving of rowmotion

Definition (V. Reiner, D. Stanton, D. White 2004)

The triple $(S, f(q), g)$ for a set S, a polynomial $f(q)$, and a bijective action g of order n, exhibits the *cyclic sieving phenomenon* when the evaluation of f at ζ^d, where $\zeta = e^{2\pi i/n}$, counts the elements of S fixed under g^d.

Cyclic sieving occurs with respect to rowmotion in the following posets:

- Minuscule posets ($\mathbf{a} \times \mathbf{b}$ is type A) (D. Rush, X. Shi)
- Minuscule poset $\times$ $\mathbf{2}$ (D. Rush, X. Shi 2013)
- Positive root posets ($\mathbf{A}_b$ is type A) (D. Armstrong, C. Stump, H. Thomas 2013)

$\rightarrow$ Cyclic sieving does not occur in general for $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$.

References on cyclic sieving

- V. Reiner, D. Stanton, D. White, What is ... cyclic sieving? Notices Amer. Math. Soc. **61** (2014), no. 2, 169–171.
- V. Reiner, D. Stanton, D. White, The cyclic sieving phenomenon. J. Combin. Theory Ser. A **108** (2004), no. 1, 17–50.
- B. Sagan, The cyclic sieving phenomenon: a survey. Surveys in combinatorics 2011, 183–233, London Math. Soc. Lecture Note Ser., **392**, Cambridge Univ. Press, Cambridge, 2011.

- Sage code: Computes the unique polynomial *of degree less than n* such that the given set and action exhibit the CSP. `http://doc.sagemath.org/html/en/reference/combinat/sage/combinat/cyclic_sieving_phenomenon.html`

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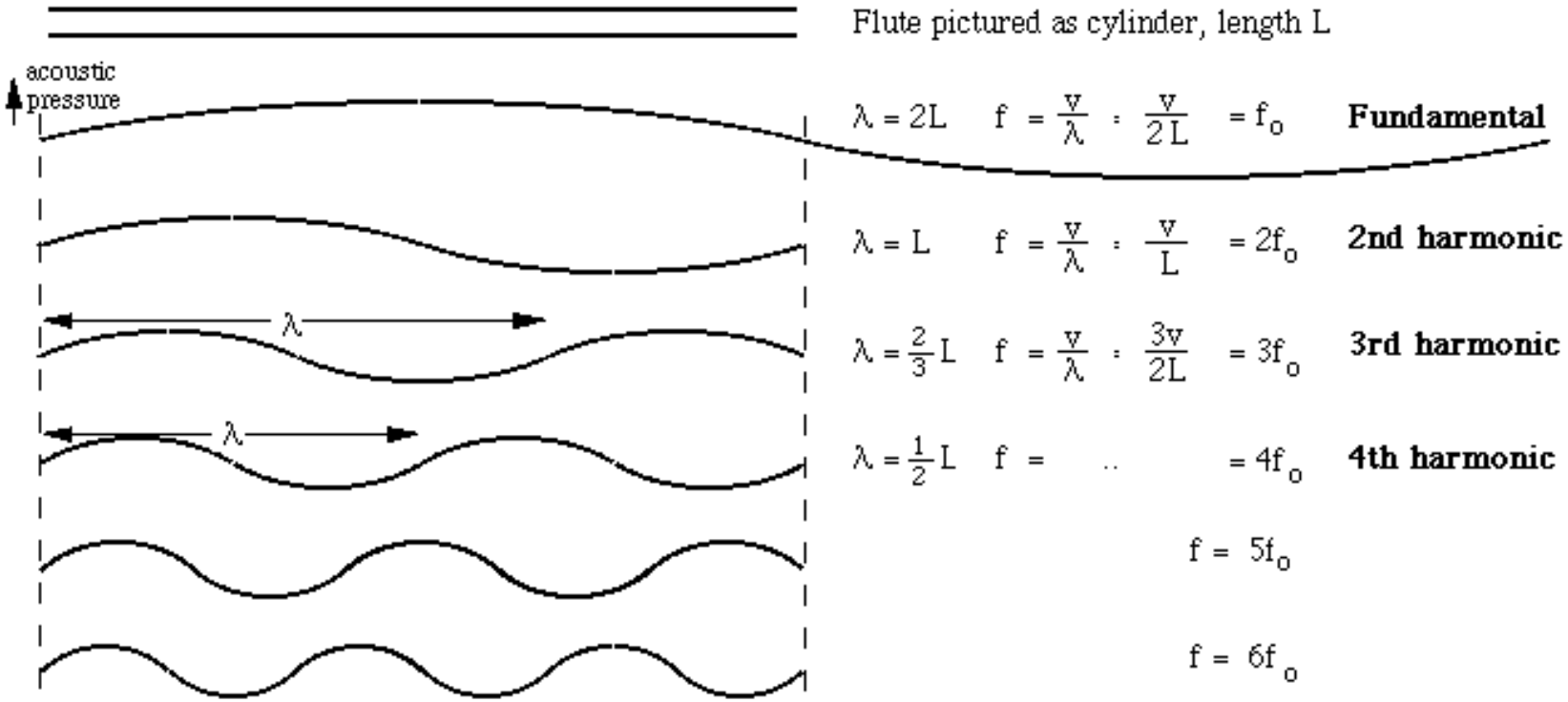
Dynamical algebraic combinatorics: Resonance

Resonance in musical instruments



http://newt.phys.unsw.edu.au/jw/fluteacoustics.html

Resonance in dynamical algebraic combinatorics

Definition (Dilks, Pechenik, Striker 2015+)

Let $G = \langle g \rangle$ be a cyclic group acting on a set X, $\mathcal{C}_\omega = \langle c \rangle$ a cyclic group of order ω acting nontrivially on a set Y, and $f : X \to Y$ a surjection. If $c \cdot f(x) = f(g \cdot x)$ for all $x \in X$, we say the triple (X, G, f) exhibits **resonance with frequency** ω.

$$\begin{tikzcd} X \arrow[r, "g"'] \arrow[d, "f"] & X \arrow[d, "f"] \\ Y \arrow[r, "c"'] & Y \end{tikzcd}$$

Dynamical algebraic combinatorics: Resonance

Promotion and rowmotion are conjugate actions

Theorem (N. Williams and S. 2012)

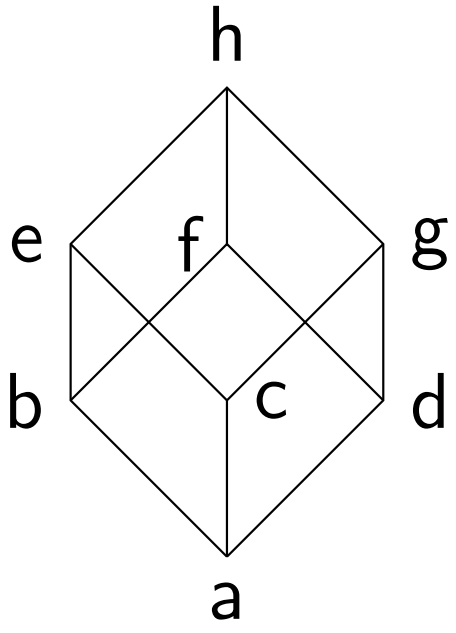
In any ranked poset, there are equivariant bijections between the order ideals under rowmotion (toggle top to bottom) and promotion (toggle left to right).

Question: What do we mean by 'left-to-right'?

Multidimensional promotion and rowmotion

Definition

A **lattice projection** of a poset P is an order and rank preserving map $\pi : P \to \mathbb{Z}^n$, where $x \leq y$ in $\mathbb{Z}^n$ if and only if the component-wise difference $y - x$ is in $\mathbb{N}^n$.



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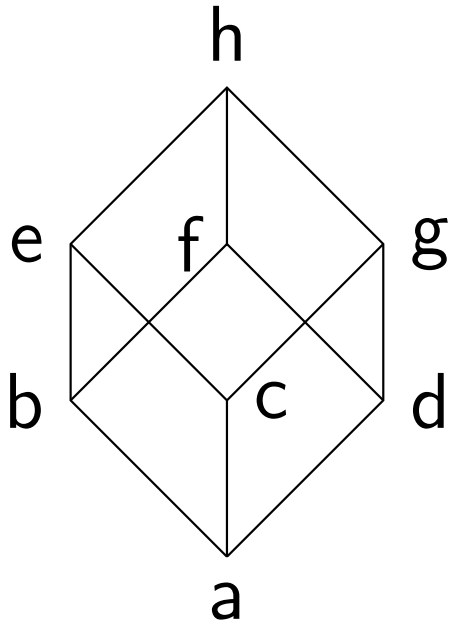

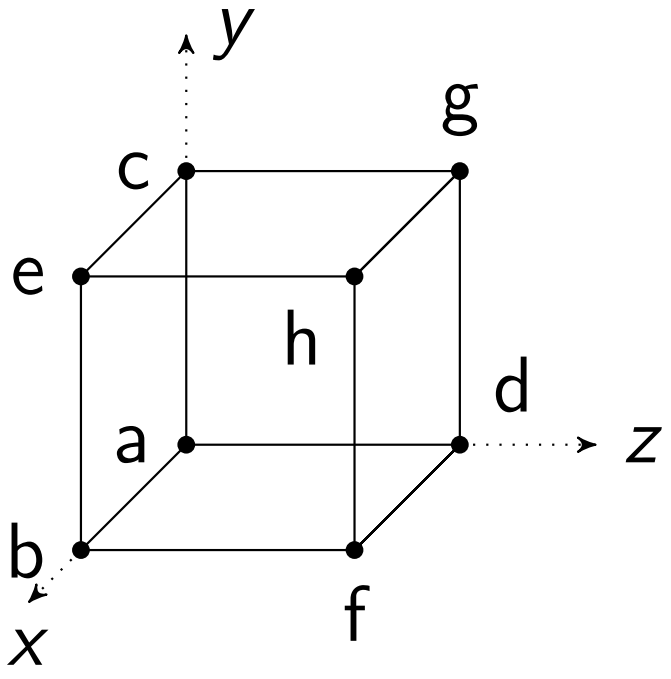

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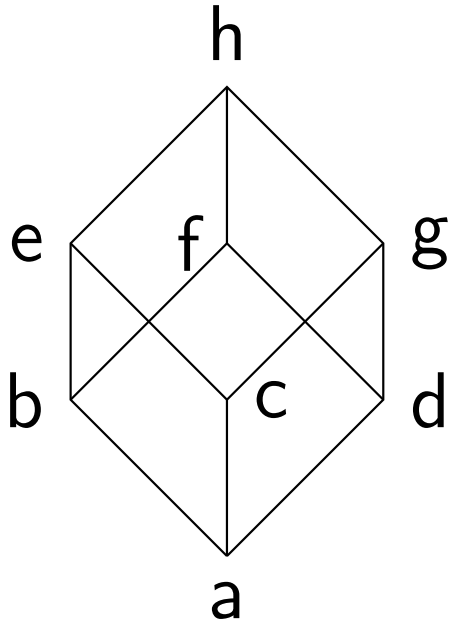
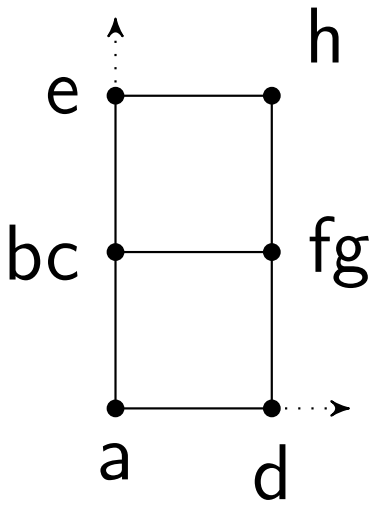
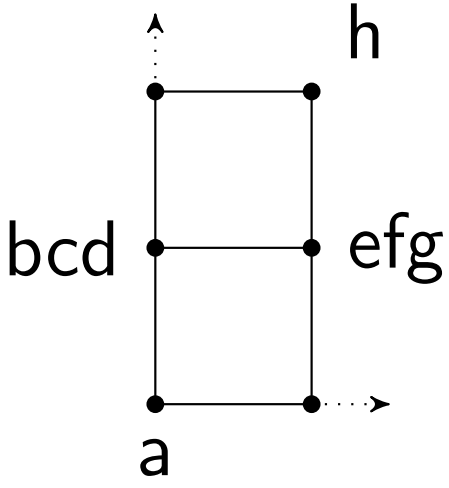






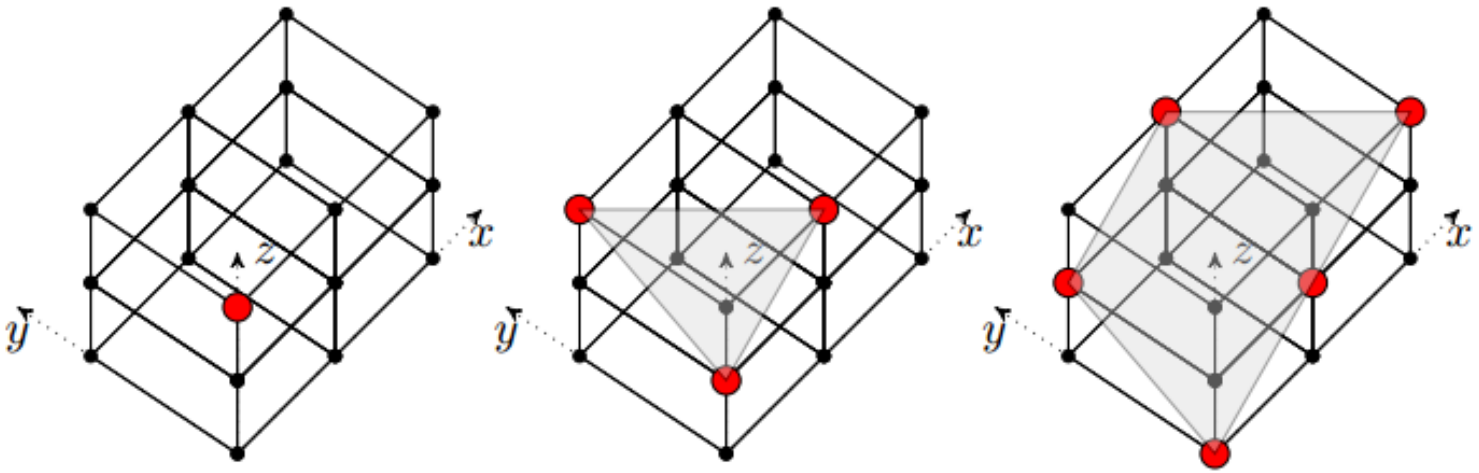
Multidimensional promotion and rowmotion

Definition

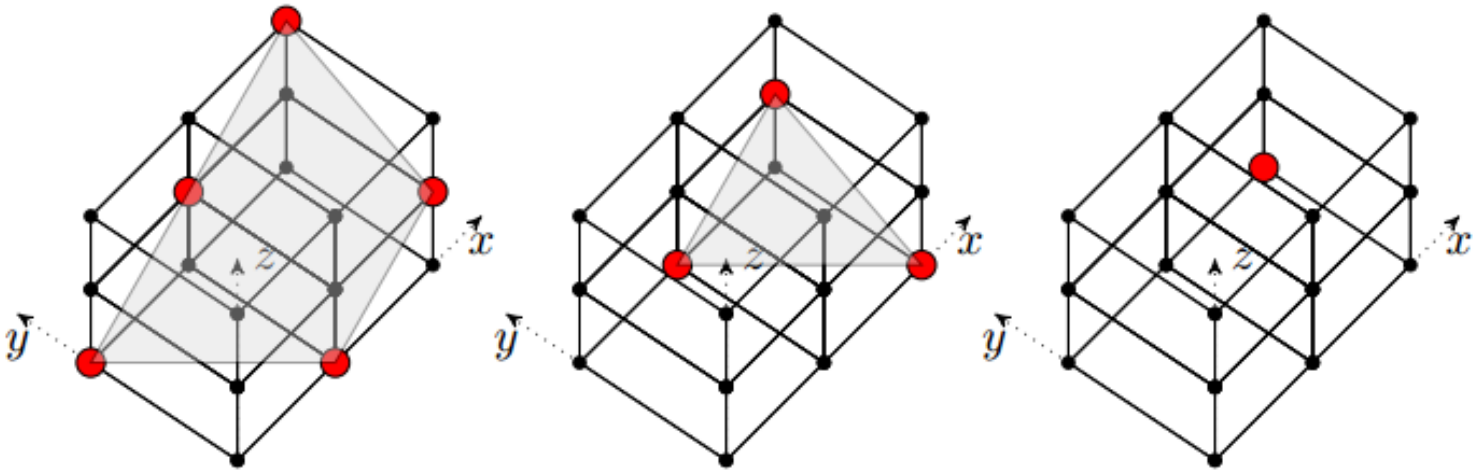
Let P be a poset with an n-dimensional lattice projection π, and let $v = (v_1, v_2, v_3, \ldots, v_n)$, where $v_j \in \{\pm 1\}$. Let $T^i_{\pi,v}$ be the product of toggles t_x for all elements x of P that lie on the affine hyperplane $\langle \pi(x), v \rangle = i$. Then define **promotion with respect to π and v** as

$$\operatorname*{Pro}_{\pi,v} = \ldots T^{-2}_{\pi,v} T^{-1}_{\pi,v} T^{0}_{\pi,v} T^{1}_{\pi,v} T^{2}_{\pi,v} \ldots.$$

$$\mathrm{Pro}_{\mathrm{id},(1,1,-1)} = T^{-2}_{\pi,v} T^{-1}_{\pi,v} T^{0}_{\pi,v} T^{1}_{\pi,v} T^{2}_{\pi,v} T^{3}_{\pi,v}$$



$x + y - z = -2$ $\quad$ $x + y - z = -1$ $\quad$ $x + y - z = 0$



$x + y - z = 1$ $\quad$ $x + y - z = 2$ $\quad$ $x + y - z = 3$

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Proposition

For any finite ranked poset P and lattice projection π, $\mathrm{Pro}_{\pi,(1,1,\ldots,1)} = \mathrm{Row}$.

Multidimensional promotion and rowmotion

Theorem (K. Dilks, O. Pechenik, S. 2017)

Let P be a finite poset with an n-dimensional lattice projection π. Let $v = (v_1, v_2, v_3, \ldots, v_n)$ and $w = (w_1, w_2, w_3, \ldots, w_n)$, where $v_j, w_j \in \{\pm 1\}$. Then there is an equivariant bijection between $J(P)$ under $\mathrm{Pro}_{\pi,v}$ and $J(P)$ under $\mathrm{Pro}_{\pi,w}$.

Rowmotion and $2^n - 1$ other promotions have the same orbit structure!

Multidimensional promotion and rowmotion

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Proof.

Similar argument as in [N. Williams and S. 2012]. □

Increasing tableaux

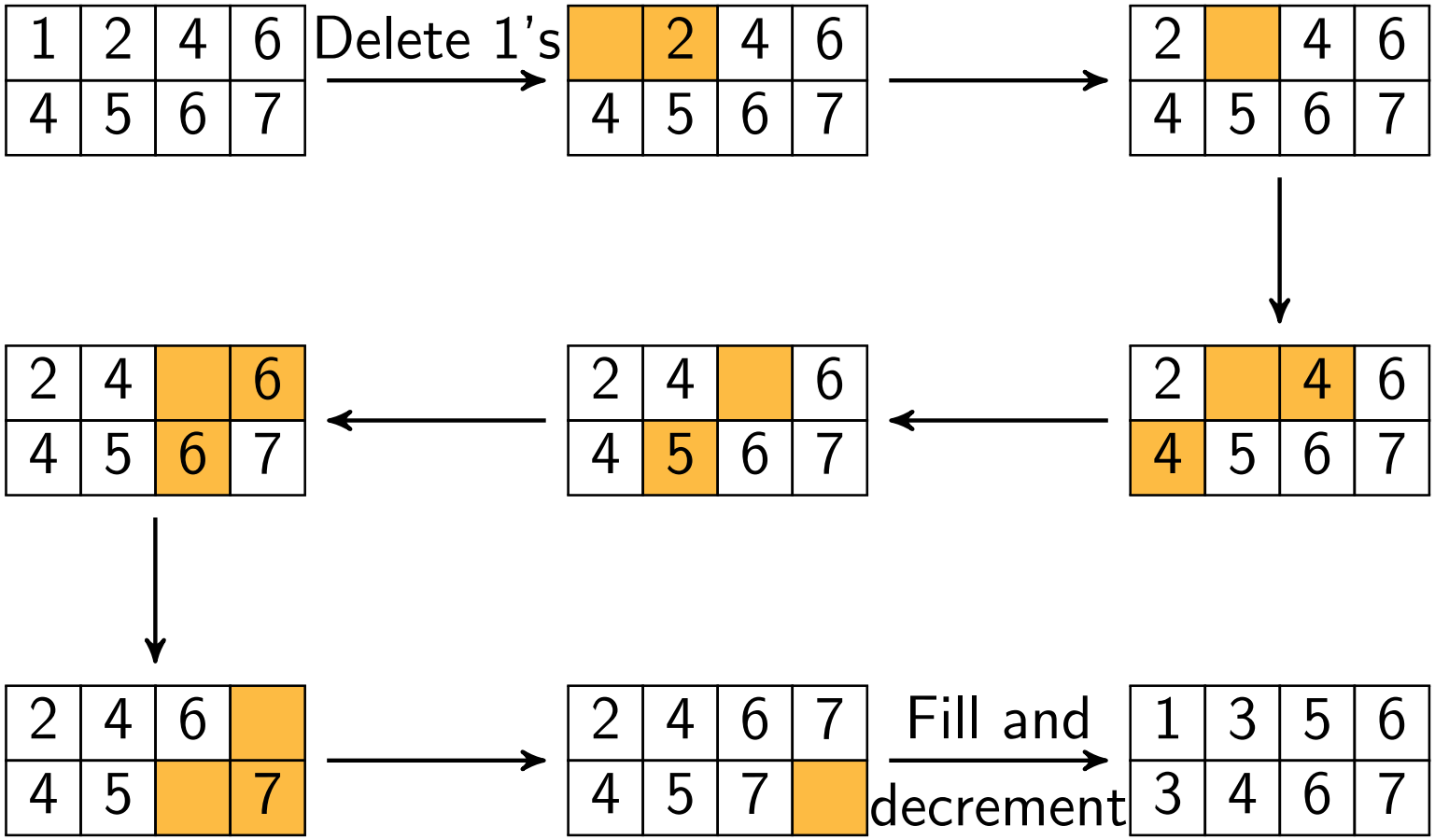
Definition

An **increasing tableau** of shape λ is a filling of a partition shape λ with positive integers so that labels strictly increase from left to right across rows and from top to bottom down columns. Let $\mathrm{Inc}^q(\lambda)$ denote the set of increasing tableaux of shape λ with entries at most q.

An increasing tableau in $\mathrm{Inc}^{10}(4,4,4,2)$:

1	4	5	8
2	5	7	9
6	7	9	10
8	10		

K-Promotion on an increasing tableau



Resonance of K-promotion

Theorem (K. Dilks, O. Pechenik, S. 2017)

$(\mathrm{Inc}^q(\lambda), \langle K\text{-Pro}\rangle, \mathrm{Con})$ *exhibits resonance with frequency q, that is, there is a projection from an increasing tableau to its binary content vector such that K-promotion maps to a cyclic shift.*

$$\begin{tikzcd}
\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 7 \\ \hline 3 & 5 & 6 & 8 \\ \hline 5 & 7 & 8 & 10 \\ \hline 7 & 9 & 10 & 12 \\ \hline \end{array} \arrow[r, "K\text{-Promotion}"] \arrow[d, "\mathrm{Con}"] & \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 & 7 & 9 \\ \hline 4 & 6 & 9 & 11 \\ \hline 6 & 8 & 11 & 12 \\ \hline \end{array} \arrow[d, "\mathrm{Con}"] \\
(1,1,1,1,1,1,1,1,1,1,0,1) \arrow[r, "\text{Cyclic shift}"] & (1,1,1,1,1,1,1,1,1,0,1,1)
\end{tikzcd}$$

K-Promotion on as a product of involutions

Proposition (K. Dilks, O. Pechenik, S. 2017)

K-Promotion is the product $\prod_i K\text{-BK}_i$, where $K\text{-BK}_i$ increments i and/or decrements $i+1$ **when possible**.

1	2	4	6
4	5	6	7

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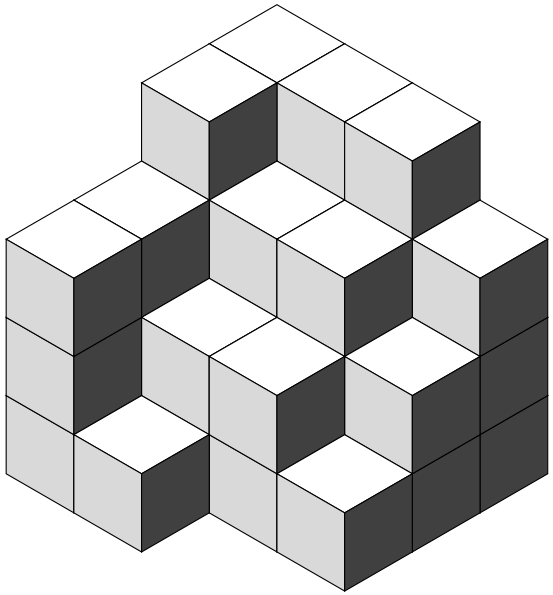
1	3	5	6
3	4	6	7

A bijection between increasing tableaux $\mathrm{Inc}^{a+b+c-1}(a \times b)$ and plane partitions $J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$

$P =$



Project to bottom face $\longrightarrow$

4	4	4	3
4	3	3	2
3	2	2	1
3	1	0	0

Rotate 180°

0	0	1	3
1	2	2	3
2	3	3	4
3	4	4	4

Add 1+rank $\longrightarrow$

1	2	4	7
3	5	6	8
5	7	8	10
7	9	10	11

$= \Psi_3(P)$

An **equivariant** bijection between $\mathrm{Inc}^{a+b+c-1}(a \times b)$ and $J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$

Theorem (K. Dilks, O. Pechenik, S. 2017)

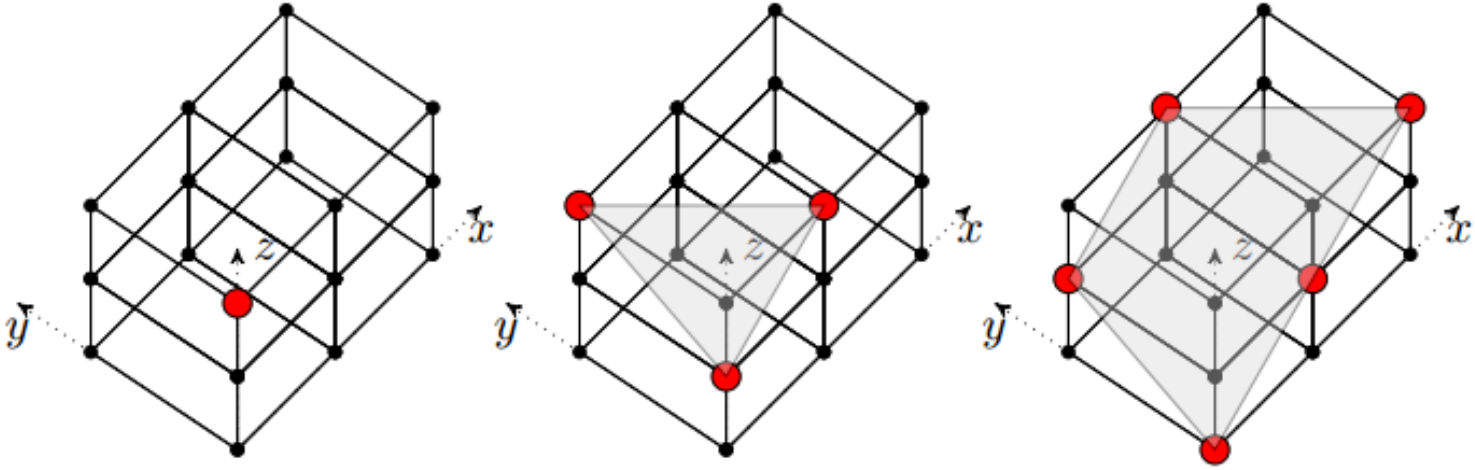
$J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$ under Row is in equivariant bijection with $\mathrm{Inc}^{a+b+c-1}(a \times b)$ under K-Pro.

Lemma (K. Dilks, O. Pechenik, S. 2017)

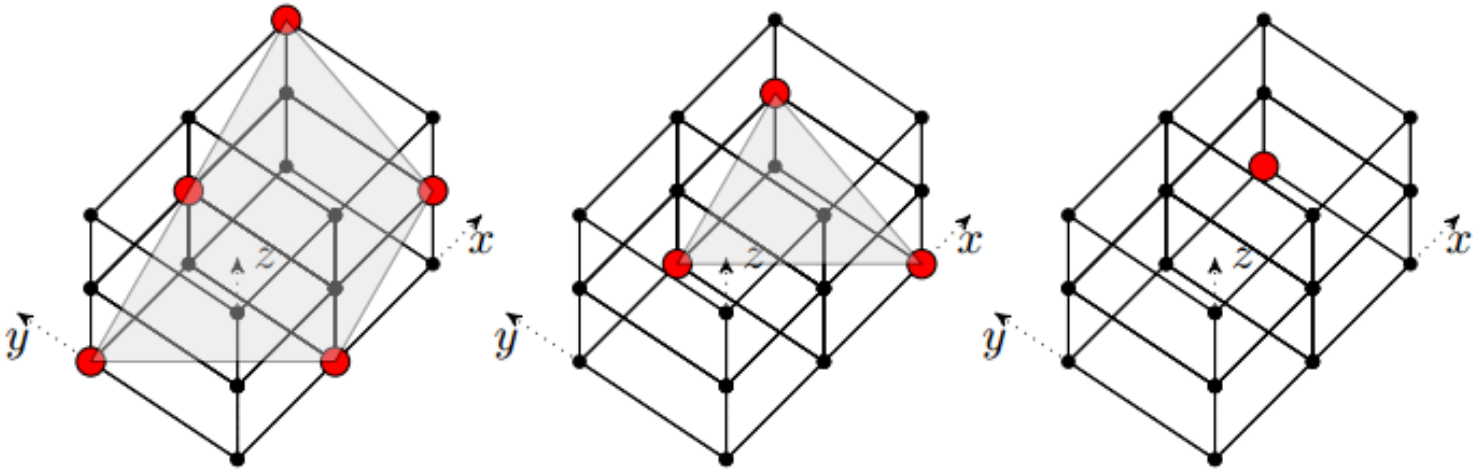
Ψ_3 intertwines $\mathrm{Pro}_{\mathrm{id},(1,1,-1)}$ and K-Pro.

$$\begin{tikzcd}
J(\mathbf{a} \times \mathbf{b} \times \mathbf{c}) \arrow[r, "\Psi_3"'] \arrow[d, "\mathrm{Pro}_{\mathrm{id},(1,1,-1)}"] & \mathrm{Inc}^{a+b+c-1}(a \times b) \arrow[d, "K\text{-Pro}"] \\
J(\mathbf{a} \times \mathbf{b} \times \mathbf{c}) \arrow[r, "\Psi_3"'] & \mathrm{Inc}^{a+b+c-1}(a \times b)
\end{tikzcd}$$

Each hyperplane-toggle corresponds to a K-BK$_i$ involution



$x + y - z = -2$ $\quad$ $x + y - z = -1$ $\quad$ $x + y - z = 0$



$x + y - z = 1$ $\quad$ $x + y - z = 2$ $\quad$ $x + y - z = 3$

Resonance of rowmotion on $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$

Theorem (K. Dilks, O. Pechenik, S. 2017)

Rowmotion on order ideals of $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$ *exhibits resonance with frequency* $a + b + c - 1$*, that is, there is a projection from an order ideal in* $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$ *through the corresponding increasing tableau to its binary content vector such that rowmotion maps to a cyclic shift.*

So even though rowmotion on $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$ does not have order $a + b + c - 1$, it projects to something that does!

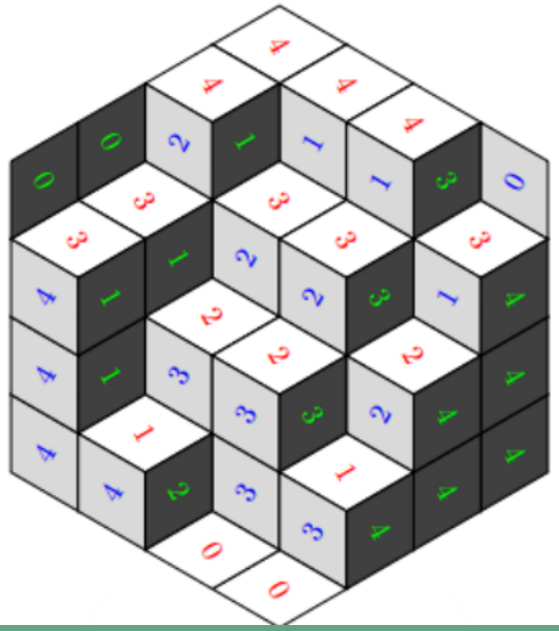
Corollaries of the equivariant bijection

Theorem (K. Dilks, O. Pechenik, S. 2017)

$J(\mathbf{a} \times \mathbf{b} \times \mathbf{c})$ under Row is in equivariant bijection with $\mathrm{Inc}^{a+b+c-1}(a \times b)$ under K-Pro.

Corollary (K. Dilks, O. Pechenik, S. 2017)

There are K-Pro-equivariant bijections between $\mathrm{Inc}^{a+b+c-1}(\lambda)$ for $\lambda = a \times b$, $a \times c$, or $b \times c$.



Corollaries of the equivariant bijection

Corollary (K. Dilks, O. Pechenik, S. 2017)

There are K-Pro-equivariant bijections between the sets $\mathrm{Inc}^{a+b+c-1}(a \times b)$, $\mathrm{Inc}^{a+b+c-1}(a \times c)$, and $\mathrm{Inc}^{a+b+c-1}(b \times c)$.

Corollary

The order of K-Pro on $\mathrm{Inc}^{a+b}(a \times b)$ is $a + b$.

Proof.

Using the tri-fold symmetry, there is a K-Pro-equivariant bijection between the sets $\mathrm{Inc}^{a+b}(a \times b)$ and $\mathrm{Inc}^{a+b}(1 \times a)$. The result is then immediate. $\square$

Corollaries of the equivariant bijection

Corollary

The order of K-Pro on $\mathrm{Inc}^{a+b+1}(a \times b)$ is $a + b + 1$.

Proof.

Using the tri-fold symmetry, there is a K-Pro-equivariant bijection between the sets $\mathrm{Inc}^{a+b+1}(a \times b)$ and $\mathrm{Inc}^{a+b+1}(2 \times a)$. The result is then immediate by a result of Pechenik on increasing tableaux or by the result of Cameron and Fon-der-Flaass on $J(\mathbf{a} \times \mathbf{b} \times \mathbf{2})$. $\square$

A conjecture

Conjecture

The order of Row *on* $J(\mathbf{a} \times \mathbf{b} \times \mathbf{3})$ *is* $a + b + 2$.

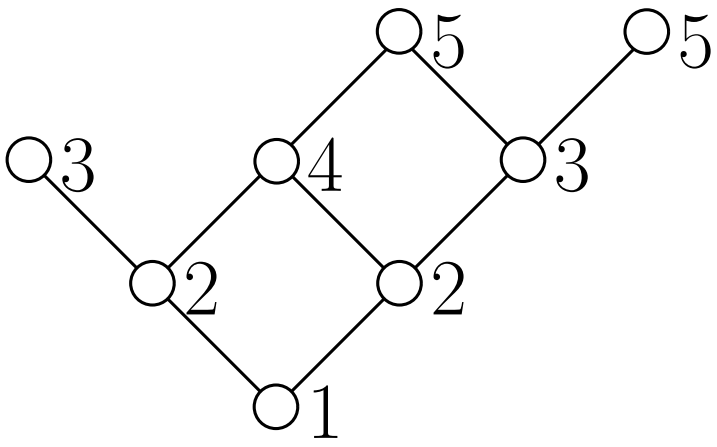
This is equivalent to the order of K-promotion being $a + b + 2$ on either $\mathrm{Inc}^{a+b+2}(a \times b)$ or $\mathrm{Inc}^{a+b+2}(3 \times a)$. We have verified conjecture for $a \leq 7$ and b arbitrary.

Dynamical algebraic combinatorics: Resonance

Promotion on increasing labelings of a poset

An increasing tableau is an **increasing labeling** of a partition shaped poset.

1	2	3	5
2	4	5	
3			



We may define generalized promotion on increasing labelings of any poset by either generalized *Bender-Knuth involutions*.

Increasing Labelings

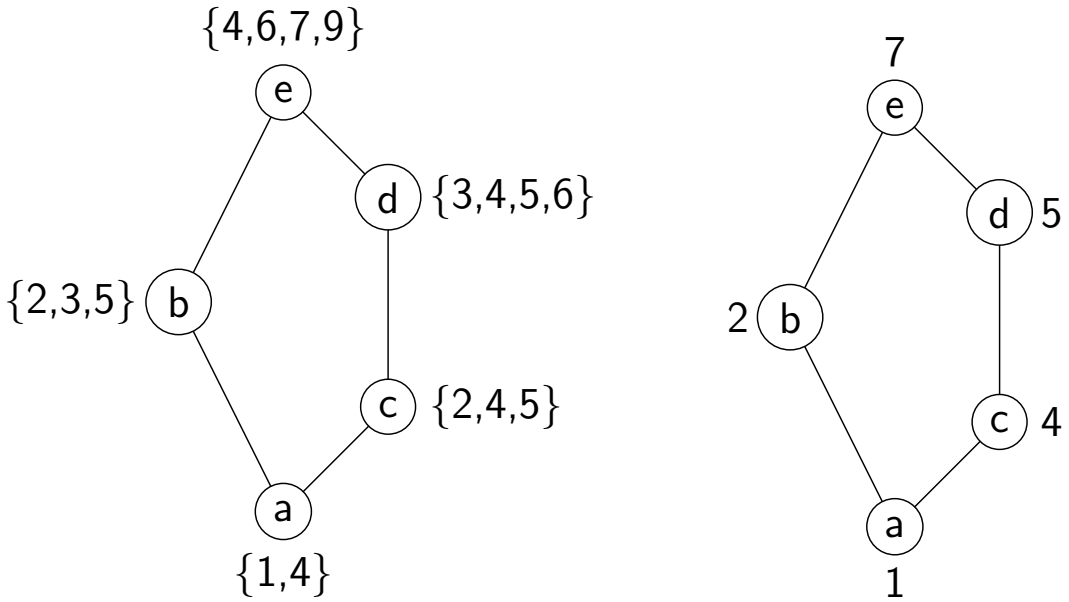
Definition

An *increasing labeling* of P is a function $f : P \to \mathbb{Z}$ such that $p_1 <_P p_2$ implies $f(p_1) < f(p_2)$.

Definition

For labeling function $R : P \mapsto \mathcal{P}(\mathbb{Z})$, let $\mathrm{Inc}^R(P)$ be the set of increasing labelings of P such that for all $p \in P$, $f(p) \in R(p)$.

$\mathrm{Inc}^R(P)$:



Increasing Labelings with entries in $\{1, \ldots, q\}$

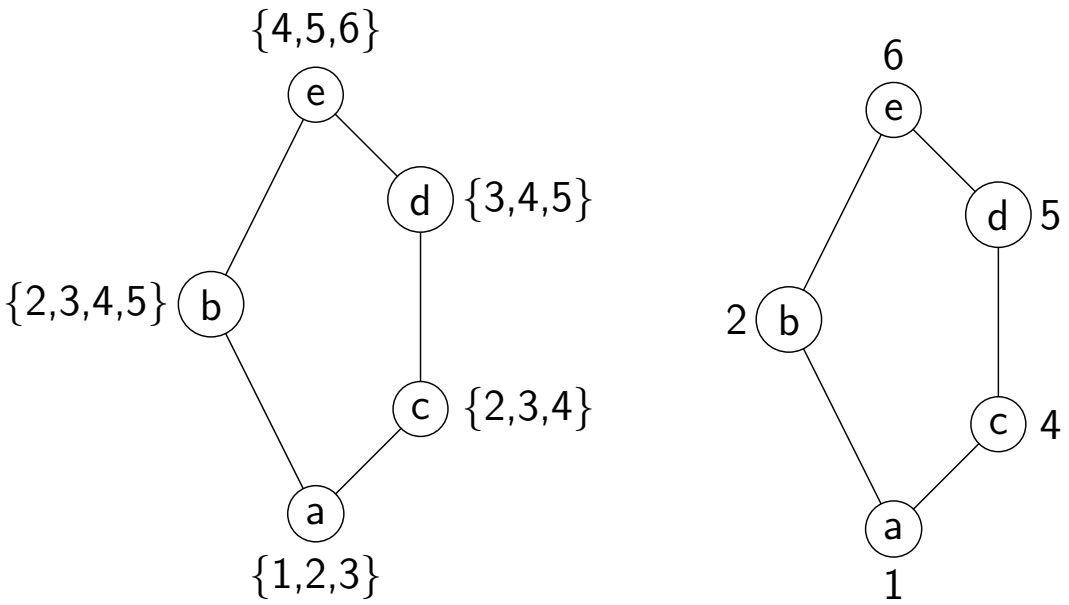
Definition

An *increasing labeling* of P is a function $f : P \to \mathbb{Z}$ such that $p_1 <_P p_2$ implies $f(p_1) < f(p_2)$.

Definition

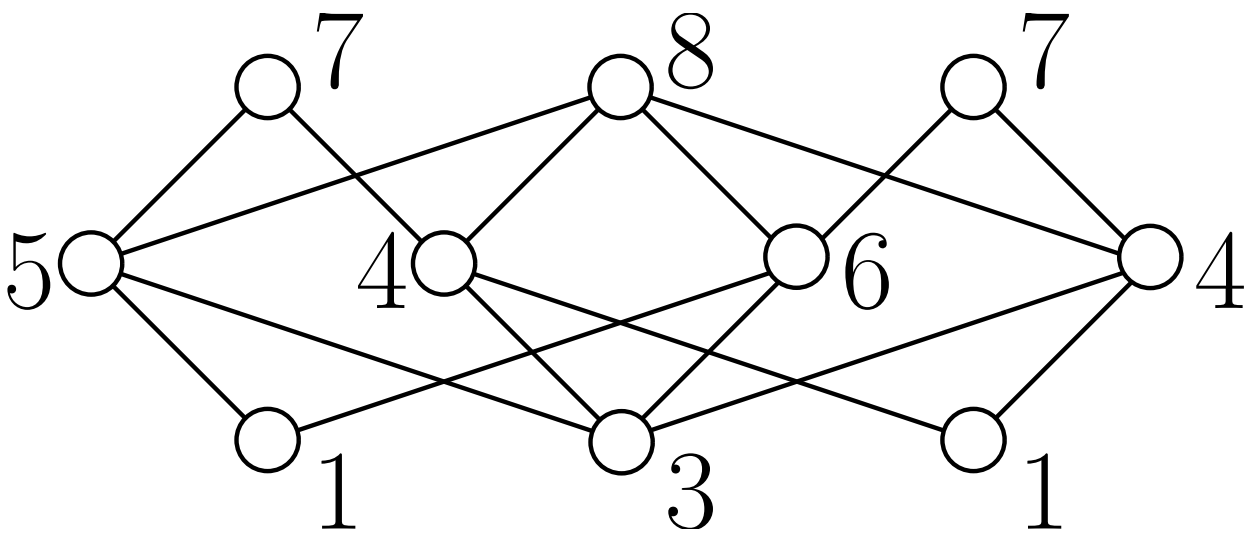
Let $\mathrm{Inc}^q(P)$ be the set of increasing labelings of P with entries in $\{1, \ldots, q\}$.

$\mathrm{Inc}^6(P)$:



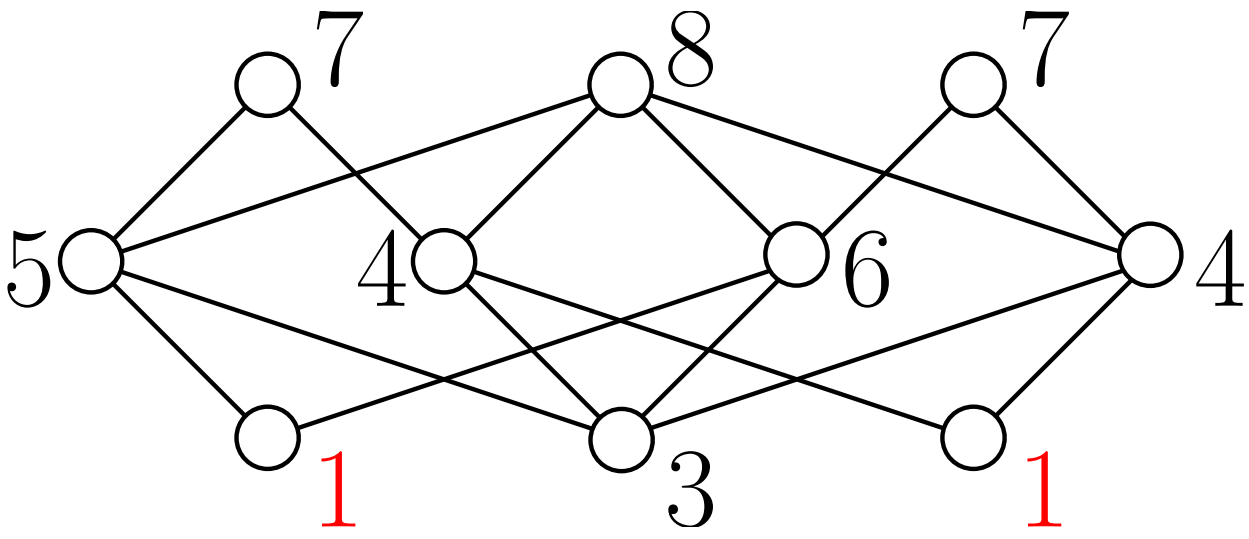
Promotion on increasing labelings $\mathrm{Inc}^q(P)$ by generalized Bender-Knuth involutions

Promotion on $\mathrm{Inc}^q(P)$ is the product $\mathrm{IncPro} = \prod_{i=1}^{q-1} \rho_i$ of the generalized Bender-Knuth involutions ρ_i, where ρ_i increments i and/or decrements $i+1$ **when possible**.



Promotion on increasing labelings $\mathrm{Inc}^q(P)$ by generalized Bender-Knuth involutions

Promotion on $\mathrm{Inc}^8(P)$ is the product $\prod_{i=1}^{7} \rho_i$ of the generalized Bender-Knuth involutions ρ_i, where ρ_1 increments 1 and/or decrements 2 when possible.



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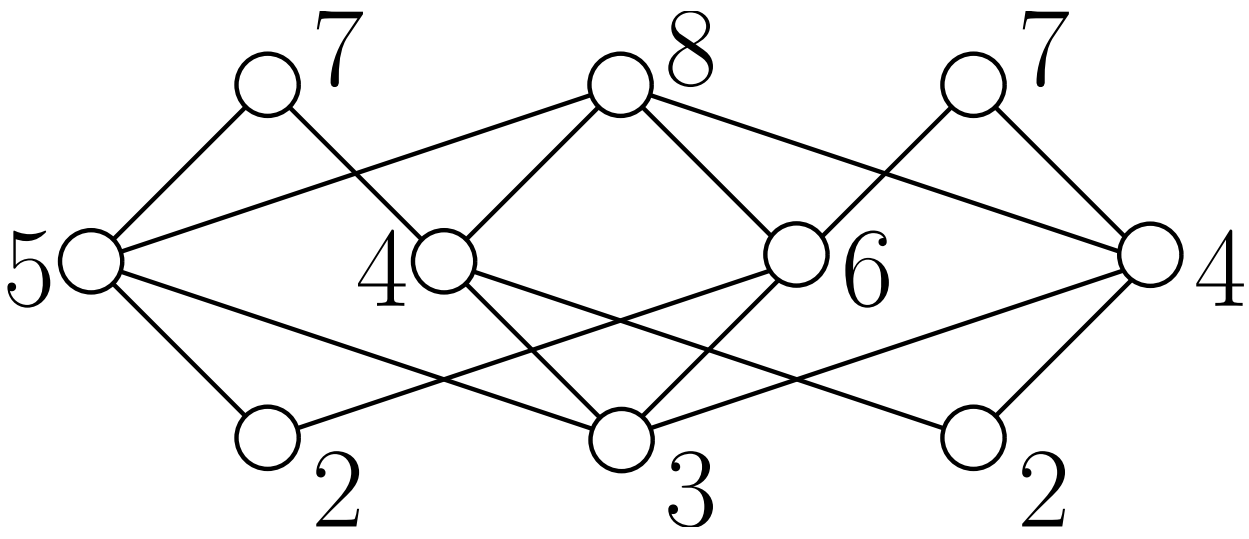
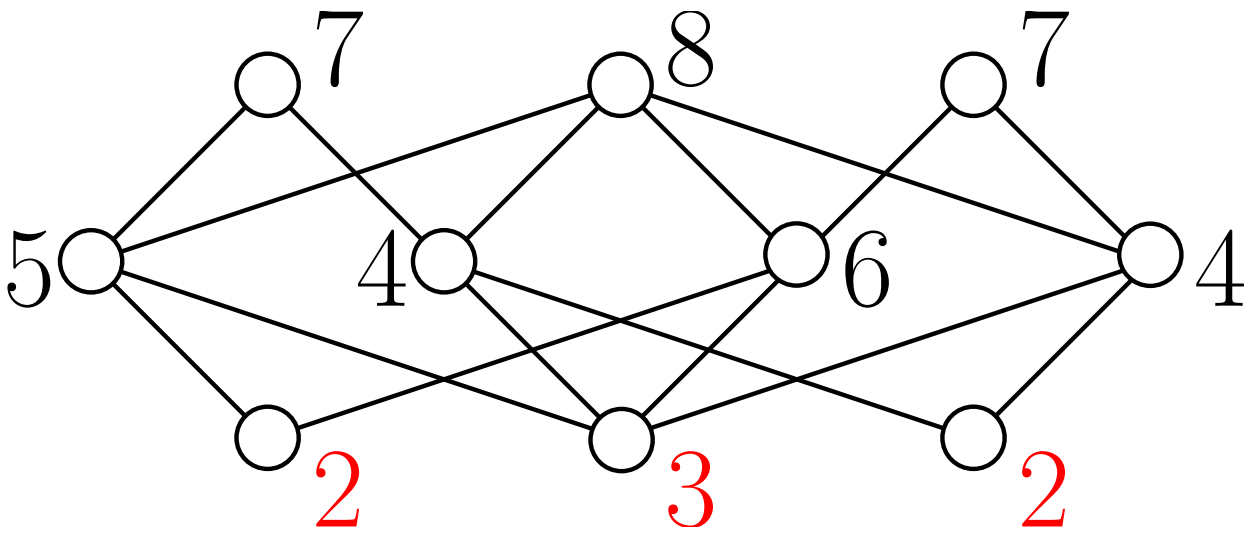


Promotion on increasing labelings $\mathrm{Inc}^q(P)$ by generalized Bender-Knuth involutions

Promotion on $\mathrm{Inc}^8(P)$ is the product $\prod_{i=1}^{7} \rho_i$ of the generalized Bender-Knuth involutions ρ_i, where ρ_2 increments 2 and/or decrements 3 when possible.



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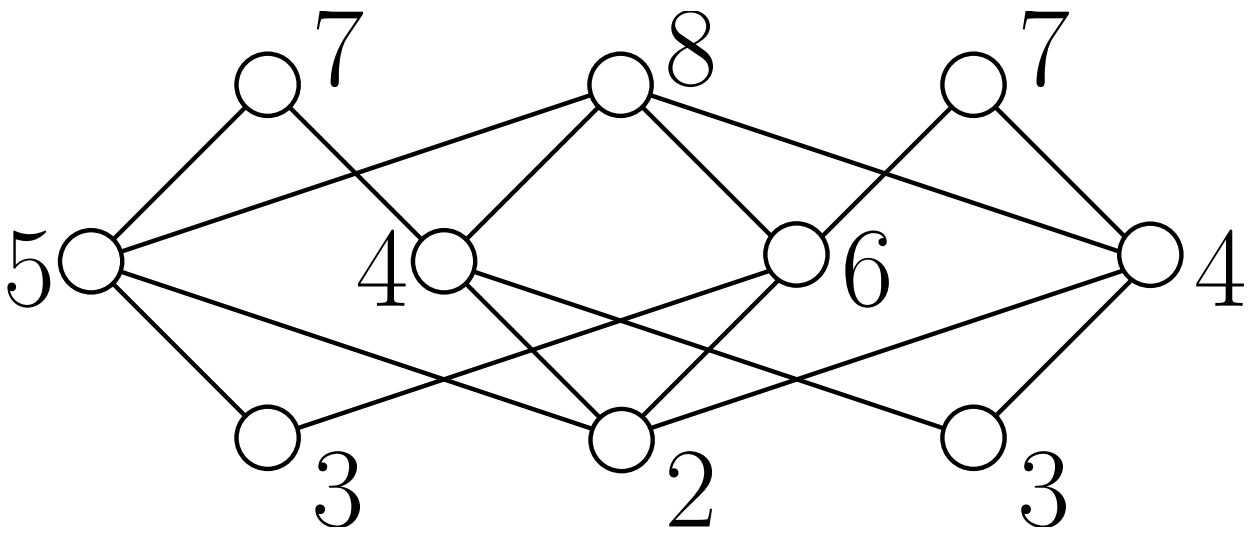
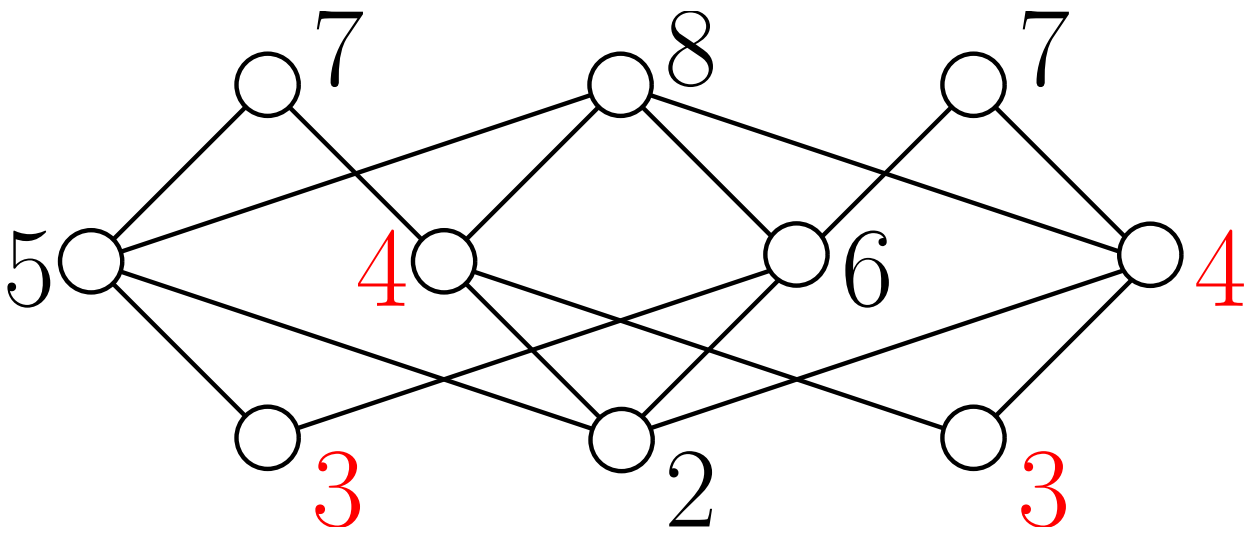


Promotion on increasing labelings $\mathrm{Inc}^q(P)$ by generalized Bender-Knuth involutions

Promotion on $\mathrm{Inc}^8(P)$ is the product $\prod_{i=1}^{7} \rho_i$ of the generalized Bender-Knuth involutions ρ_i, where ρ_3 increments 3 and/or decrements 4 when possible.



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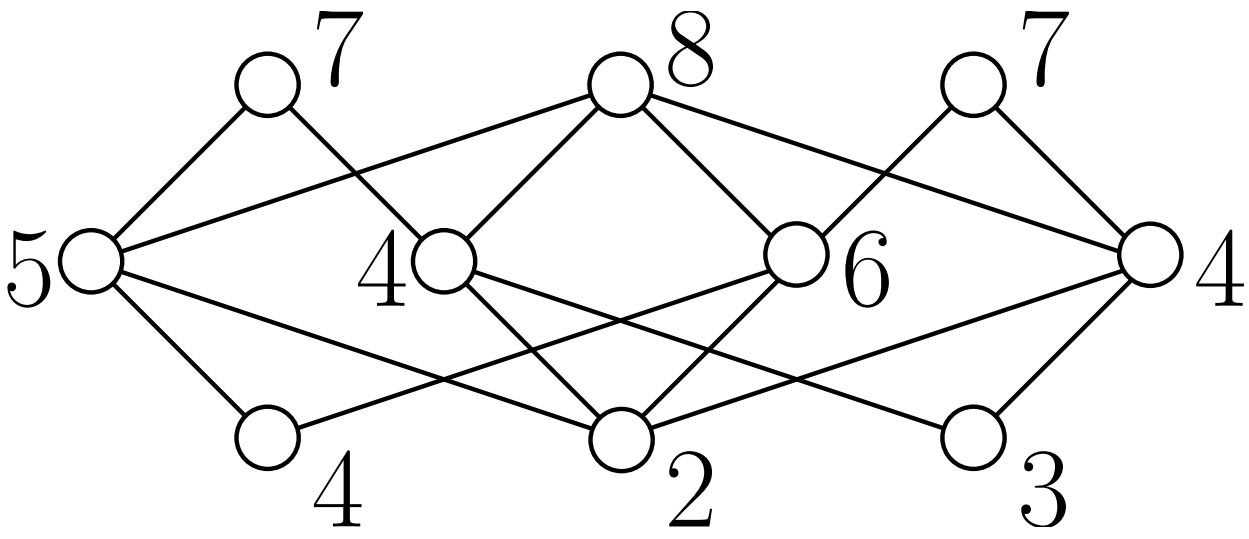
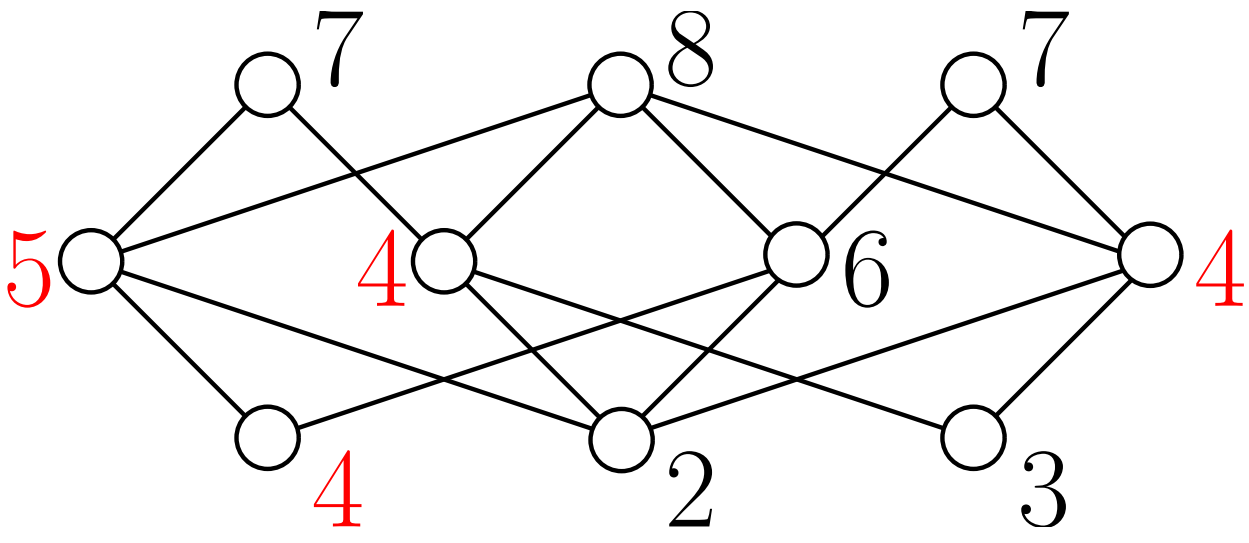


Promotion on increasing labelings $\mathrm{Inc}^q(P)$ by generalized Bender-Knuth involutions

Promotion on $\mathrm{Inc}^8(P)$ is the product $\prod_{i=1}^{7} \rho_i$ of the generalized Bender-Knuth involutions ρ_i, where ρ_4 increments 4 and/or decrements 5 when possible.



Promotion on increasing labelings $\mathrm{Inc}^q(P)$ by generalized Bender-Knuth involutions

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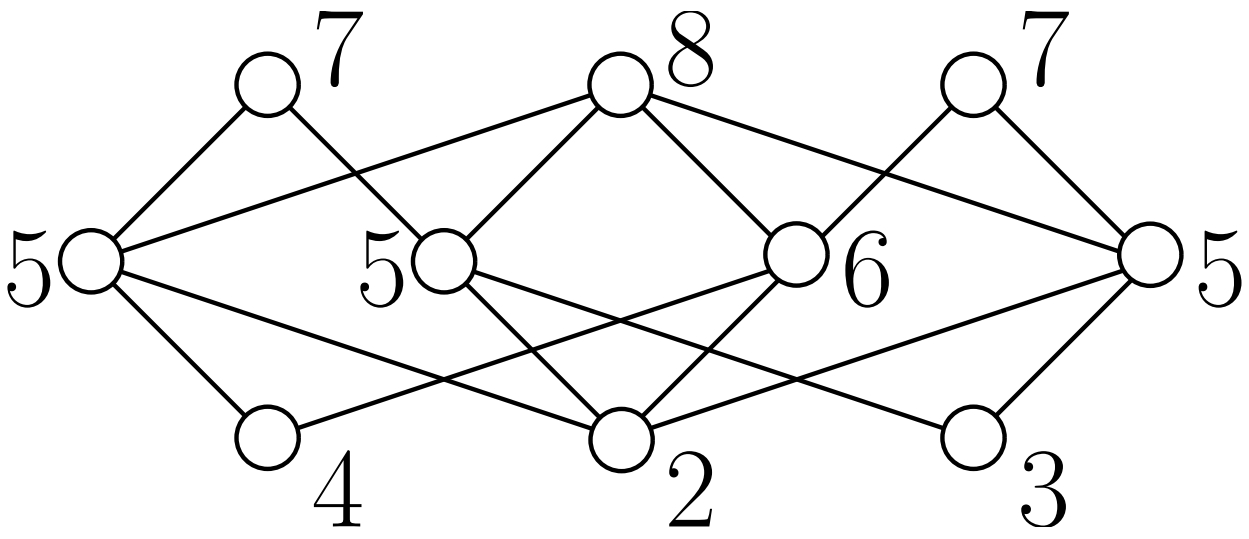
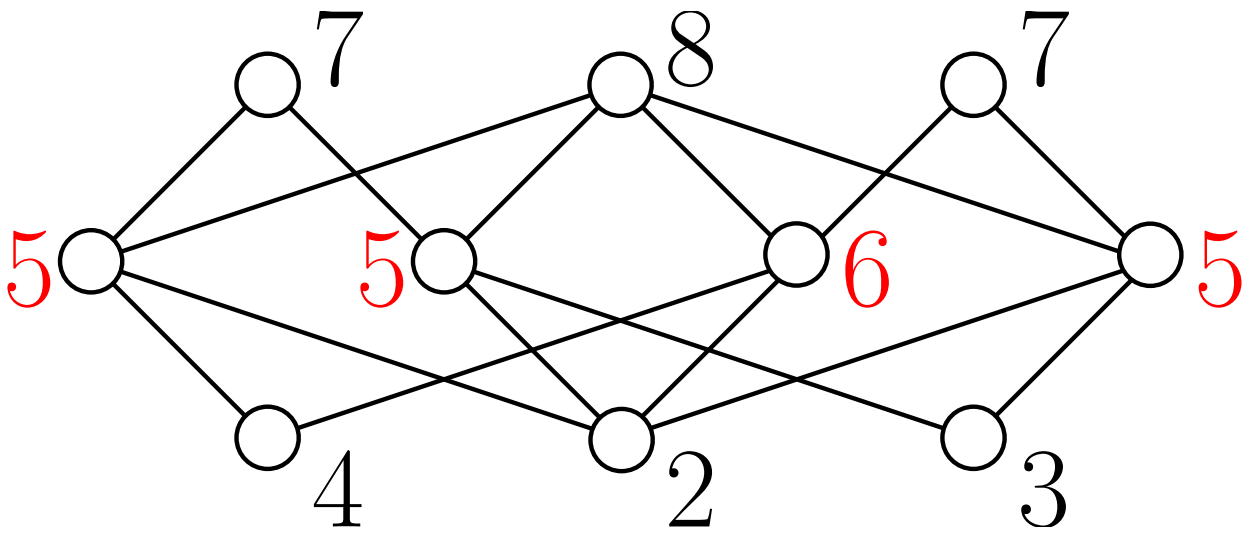


Promotion on increasing labelings $\mathrm{Inc}^q(P)$ by generalized Bender-Knuth involutions

Promotion on $\mathrm{Inc}^8(P)$ is the product $\prod_{i=1}^{7} \rho_i$ of the generalized Bender-Knuth involutions ρ_i, where ρ_5 increments 5 and/or decrements 6 when possible.



Promotion on increasing labelings $\mathrm{Inc}^q(P)$ by generalized Bender-Knuth involutions

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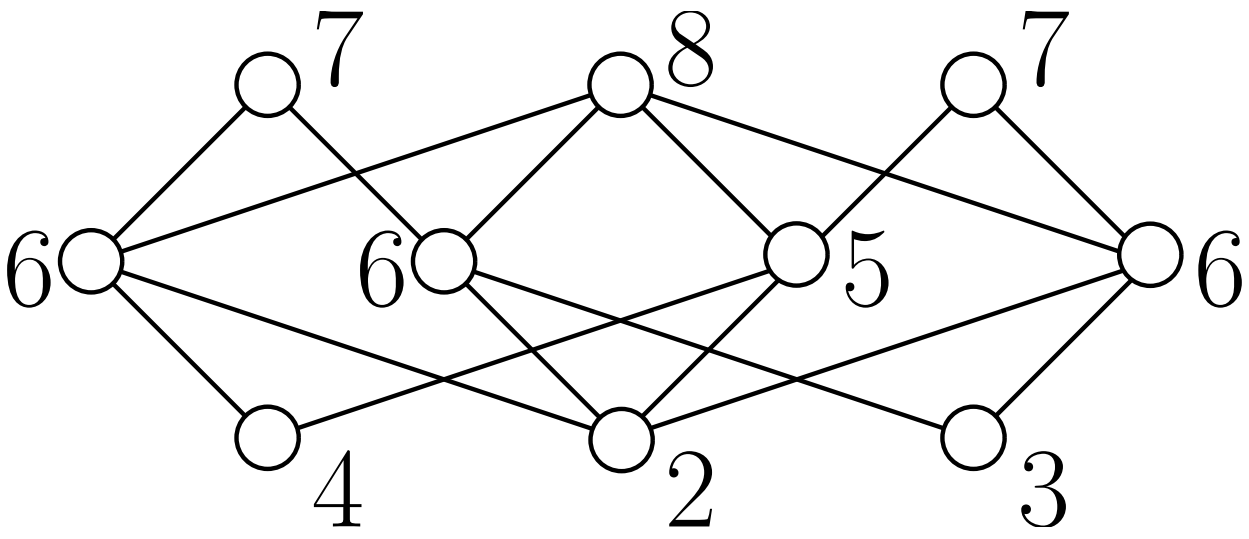
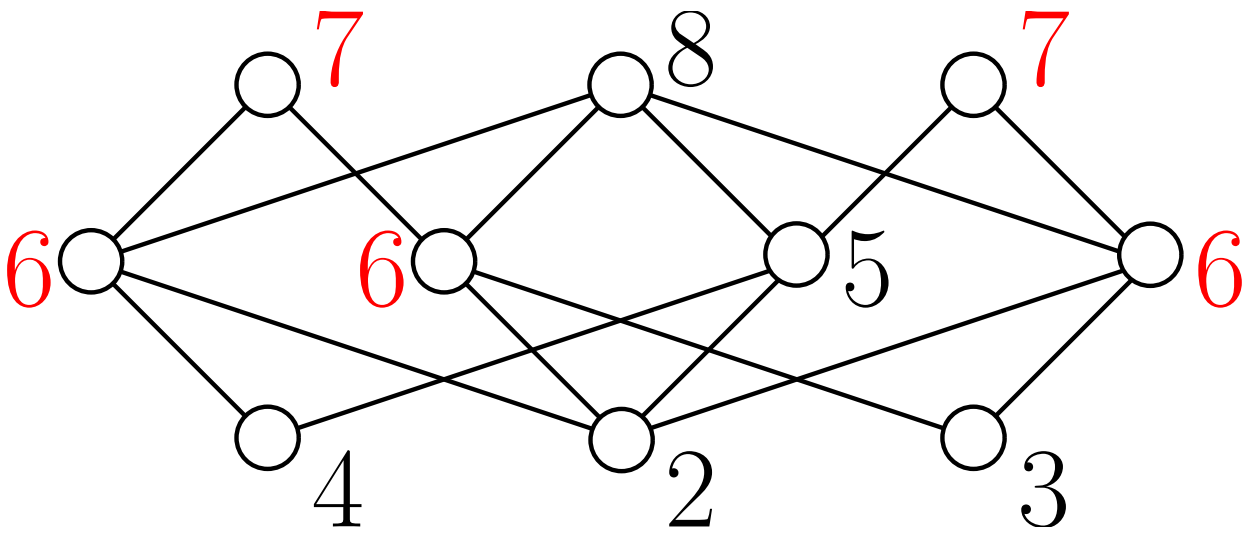


Promotion on increasing labelings $\mathrm{Inc}^q(P)$ by generalized Bender-Knuth involutions

Promotion on $\mathrm{Inc}^8(P)$ is the product $\prod_{i=1}^{7} \rho_i$ of the generalized Bender-Knuth involutions ρ_i, where ρ_6 increments 6 and/or decrements 7 when possible.



Promotion on increasing labelings $\mathrm{Inc}^q(P)$ by generalized Bender-Knuth involutions

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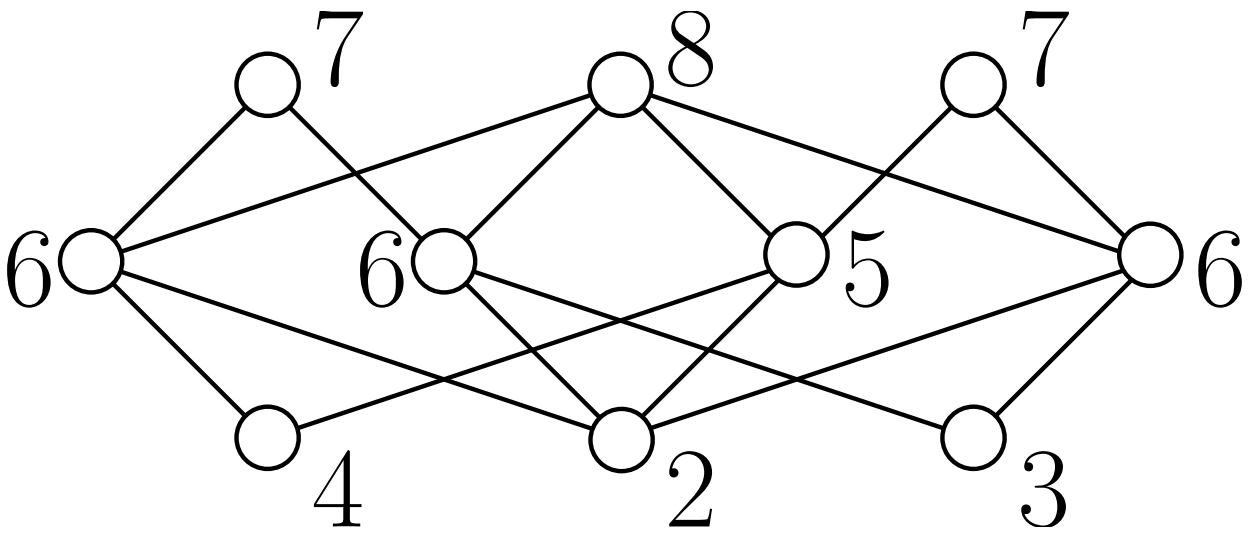
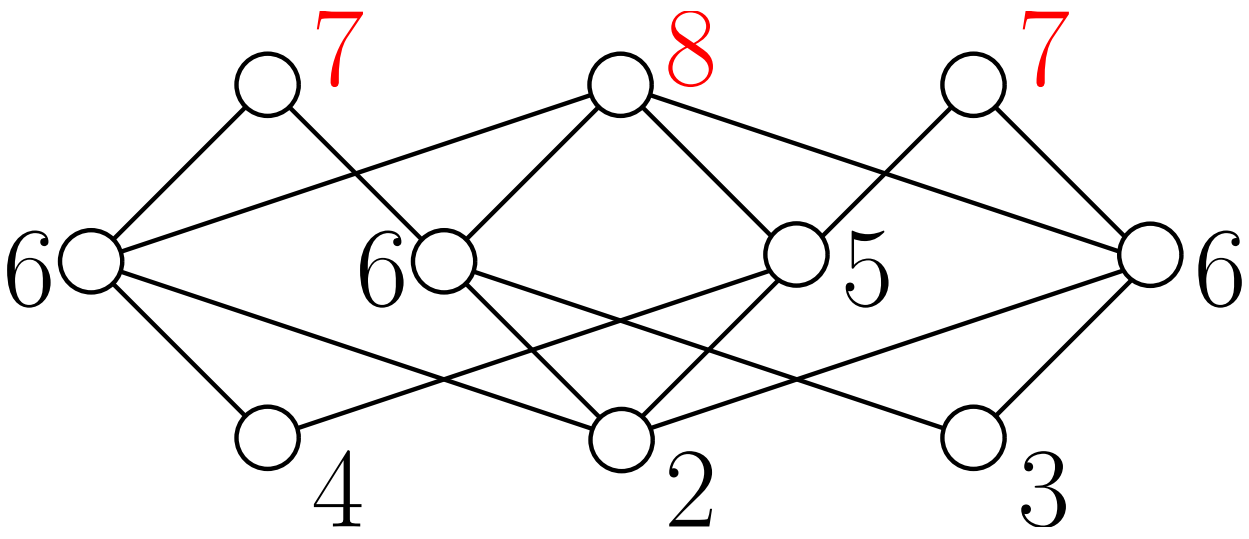


Promotion on increasing labelings $\mathrm{Inc}^q(P)$ by generalized Bender-Knuth involutions

Promotion on $\mathrm{Inc}^8(P)$ is the product $\prod_{i=1}^{7} \rho_i$ of the generalized Bender-Knuth involutions ρ_i, where ρ_7 increments 7 and/or decrements 8 when possible.



Promotion on increasing labelings $\mathrm{Inc}^q(P)$ by generalized Bender-Knuth involutions

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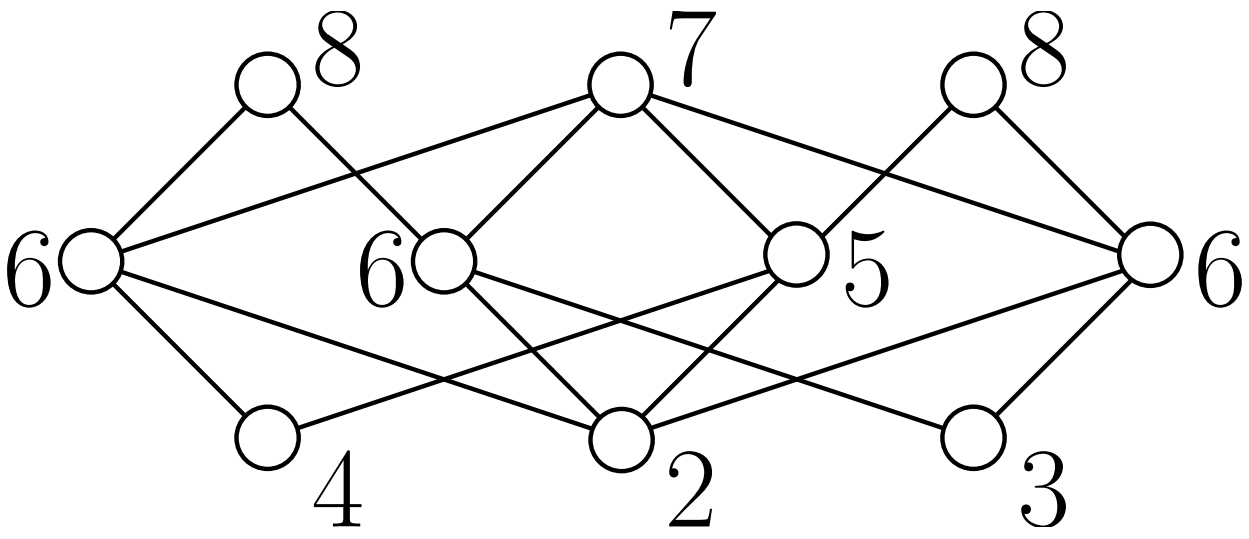


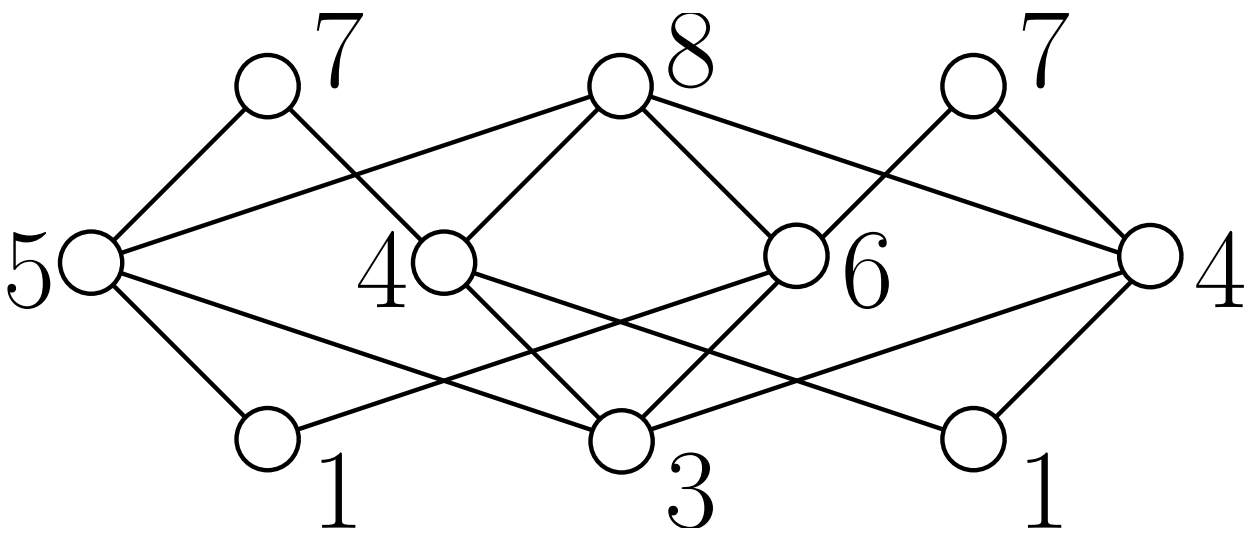
Promotion on increasing labelings $\mathrm{Inc}^R(P)$ by generalized Bender-Knuth involutions

The generalized Bender-Knuth involution ρ_i for an arbitrary restriction function R acts as follows:

- If a label is currently i, and you can increment it to next allowable label (and stay in $\mathrm{Inc}^R(P)$), then do so.
- If you can decrement a label to become i (and stay in $\mathrm{Inc}^R(P)$), then do so.
- Otherwise, do nothing.

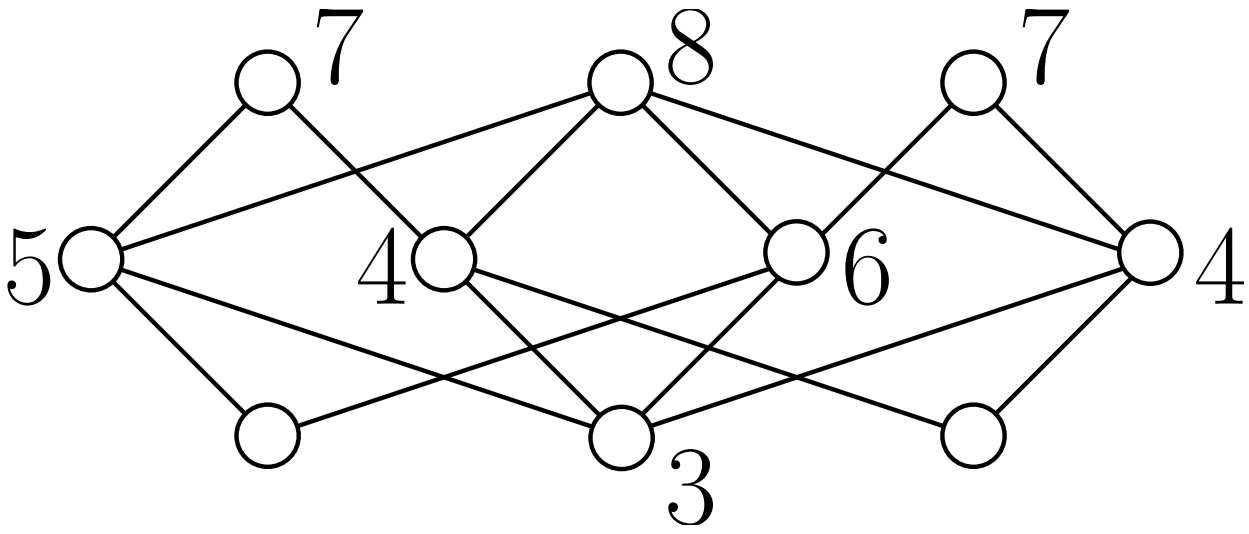
Then generalized increasing labeling promotion on $\mathrm{Inc}^R(P)$ is: $\mathsf{IncPro} = \cdots \circ \rho_2 \circ \rho_1 \circ \cdots$.

Promotion on increasing labelings $\mathrm{Inc}^q(P)$ by generalized jeu de taquin slides



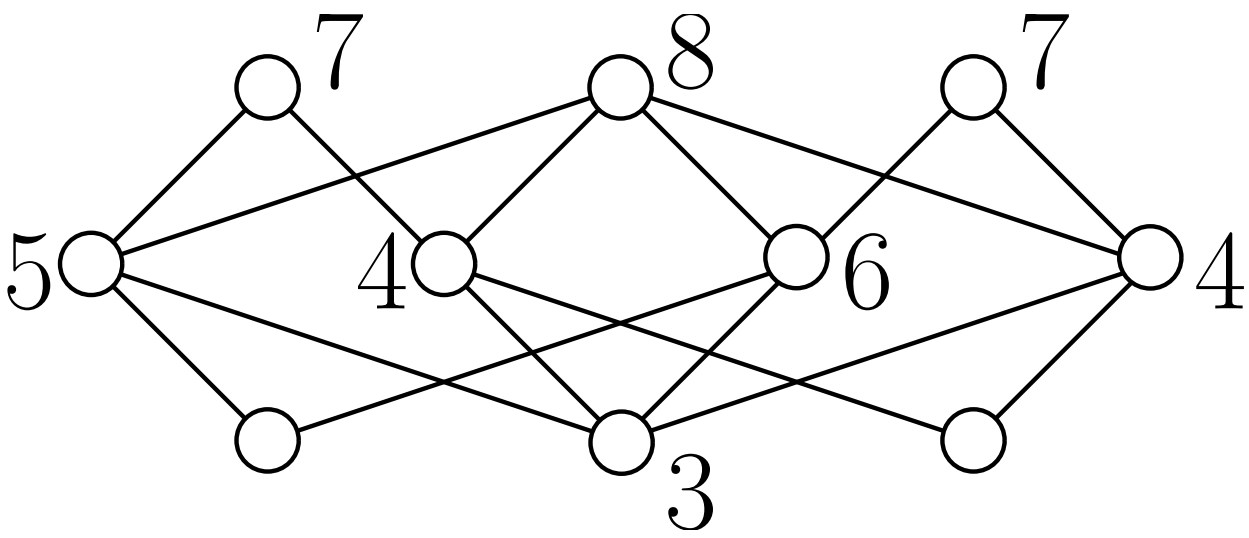
Promotion on increasing labelings $\mathrm{Inc}^q(P)$ by generalized jeu de taquin slides

Delete any 1's.



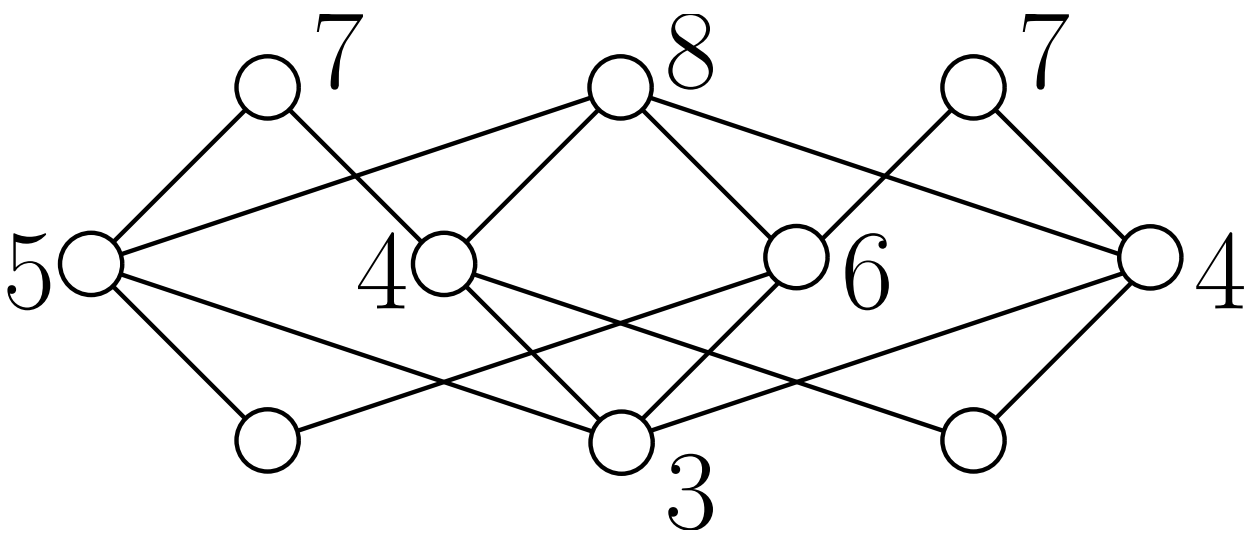
Promotion on increasing labelings $\mathrm{Inc}^q(P)$ by generalized jeu de taquin slides

Consider blank spots and 2's. Fill any blank spot that has a 2 covering it by a 2. Simultaneously, replace any 2's covering blank spots by a blank spot.



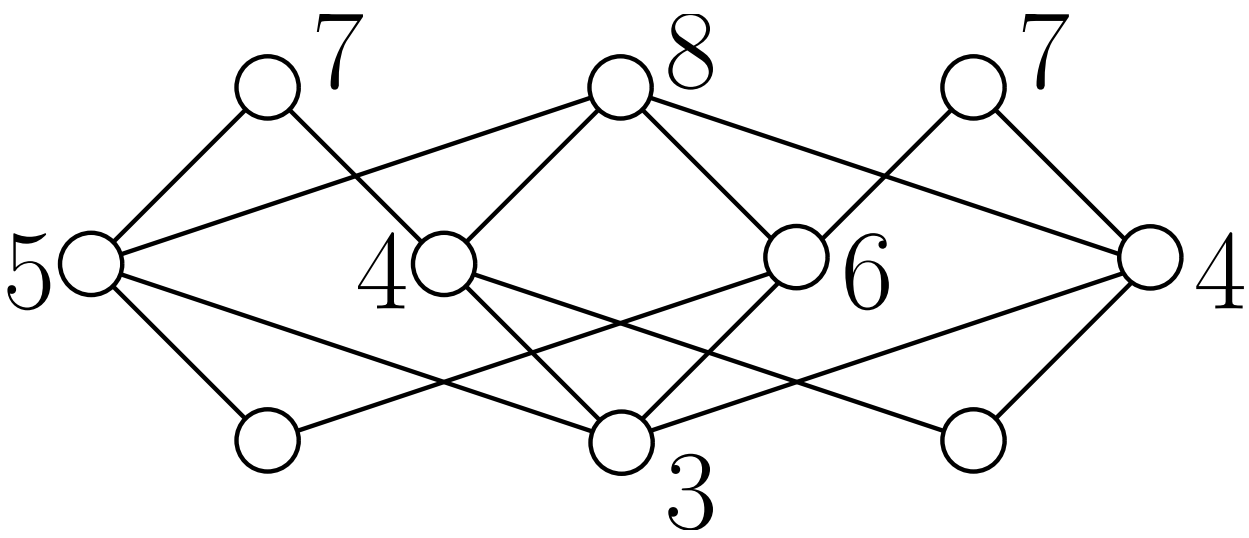
Promotion on increasing labelings $\mathrm{Inc}^q(P)$ by generalized jeu de taquin slides

Consider blank spots and 3's. Fill any blank spot that has a 3 covering it by a 3. Simultaneously, replace any 3's covering blank spots by a blank spot.



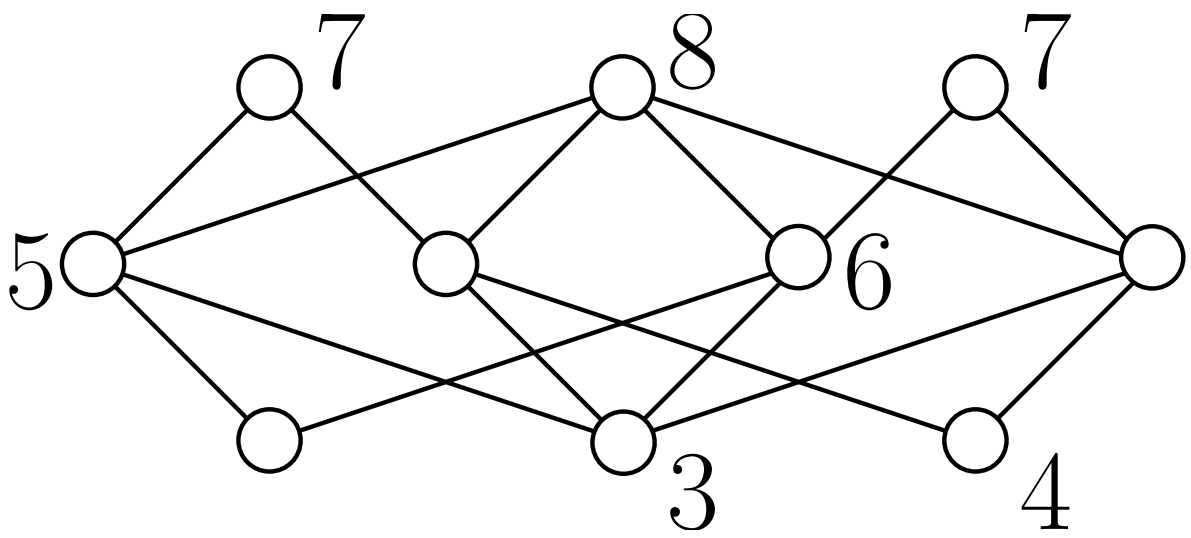
Promotion on increasing labelings $\mathrm{Inc}^q(P)$ by generalized jeu de taquin slides

Consider blank spots and 4's. Fill any blank spot that has a 4 covering it by a 4. Simultaneously, replace any 4's covering blank spots by a blank spot.



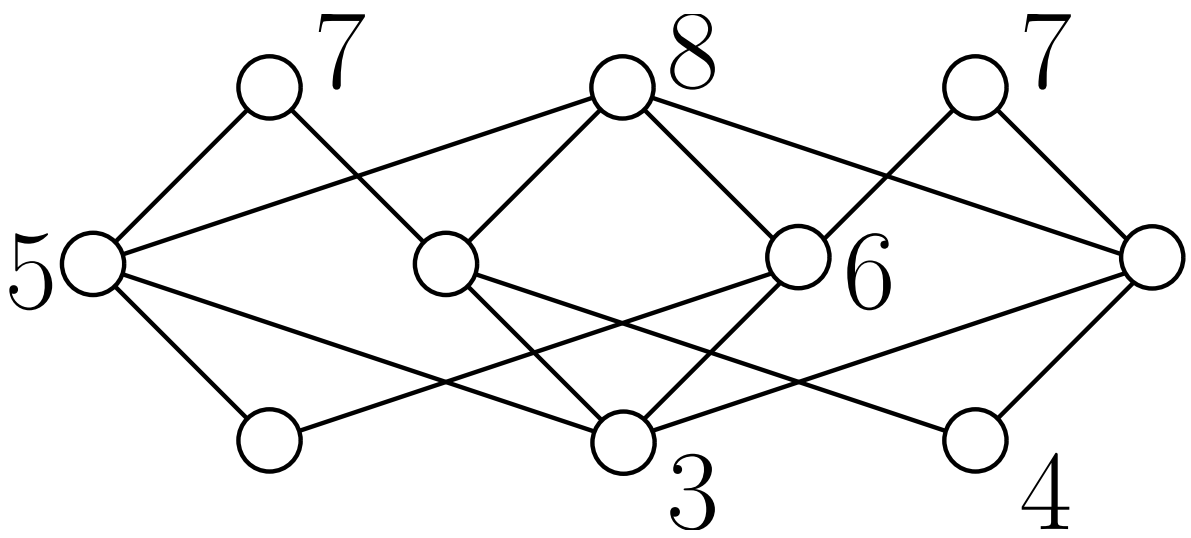
Promotion on increasing labelings $\mathrm{Inc}^q(P)$ by generalized jeu de taquin slides

Consider blank spots and 4's. Fill any blank spot that has a 4 covering it by a 4. Simultaneously, replace any 4's covering blank spots by a blank spot.



Promotion on increasing labelings $\mathrm{Inc}^q(P)$ by generalized jeu de taquin slides

Consider blank spots and 5's. Fill any blank spot that has a 5 covering it by a 5. Simultaneously, replace any 5's covering blank spots by a blank spot.



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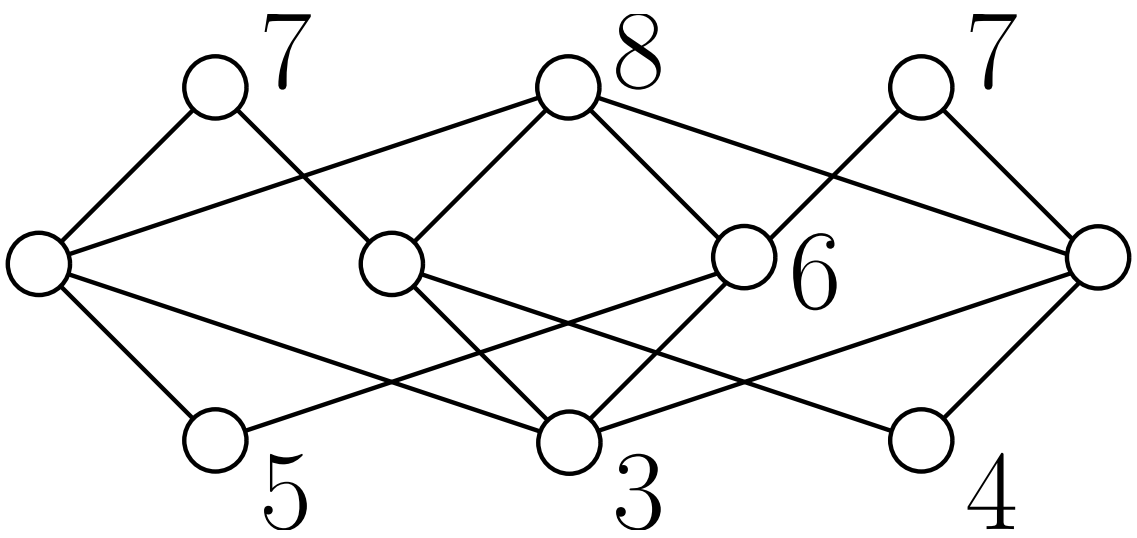
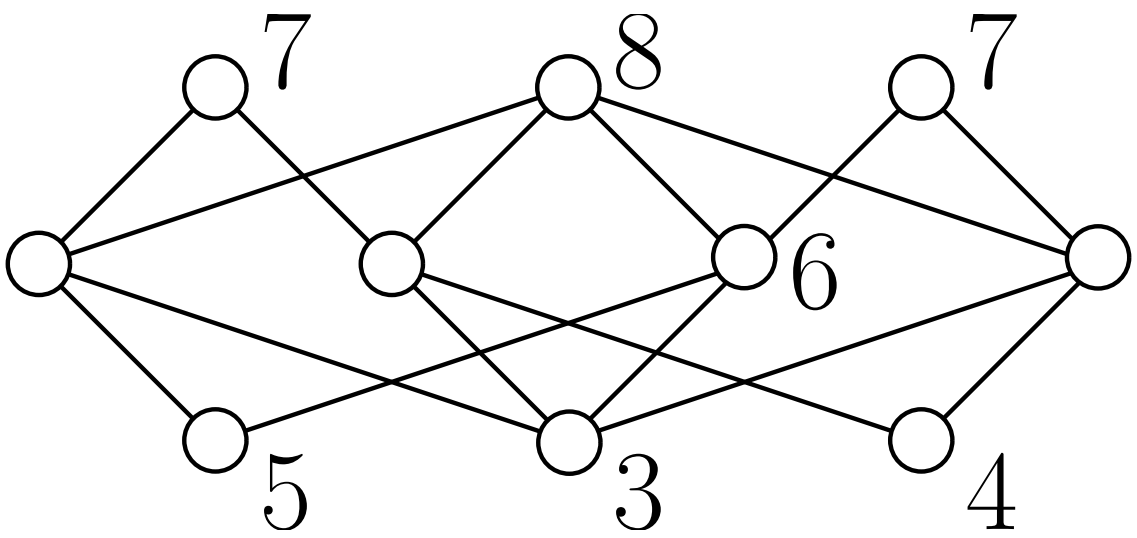


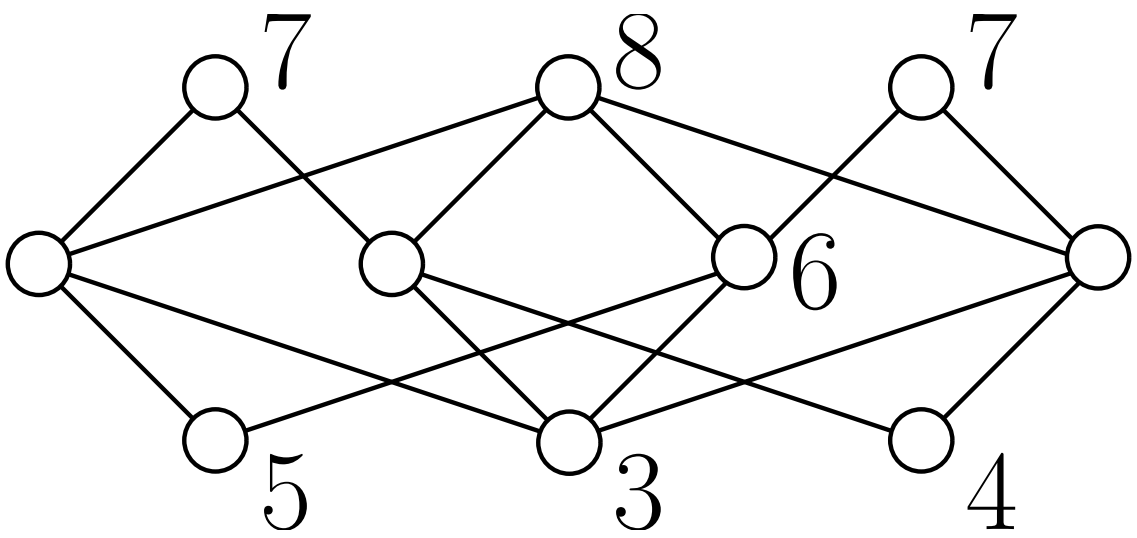
Promotion on increasing labelings $\mathrm{Inc}^q(P)$ by generalized jeu de taquin slides

Consider blank spots and 6's. Fill any blank spot that has a 6 covering it by a 6. Simultaneously, replace any 5's covering blank spots by a blank spot.



Promotion on increasing labelings $\mathrm{Inc}^q(P)$ by generalized jeu de taquin slides

Consider blank spots and 7's. Fill any blank spot that has a 7 covering it by a 7. Simultaneously, replace any 7's covering blank spots by a blank spot.



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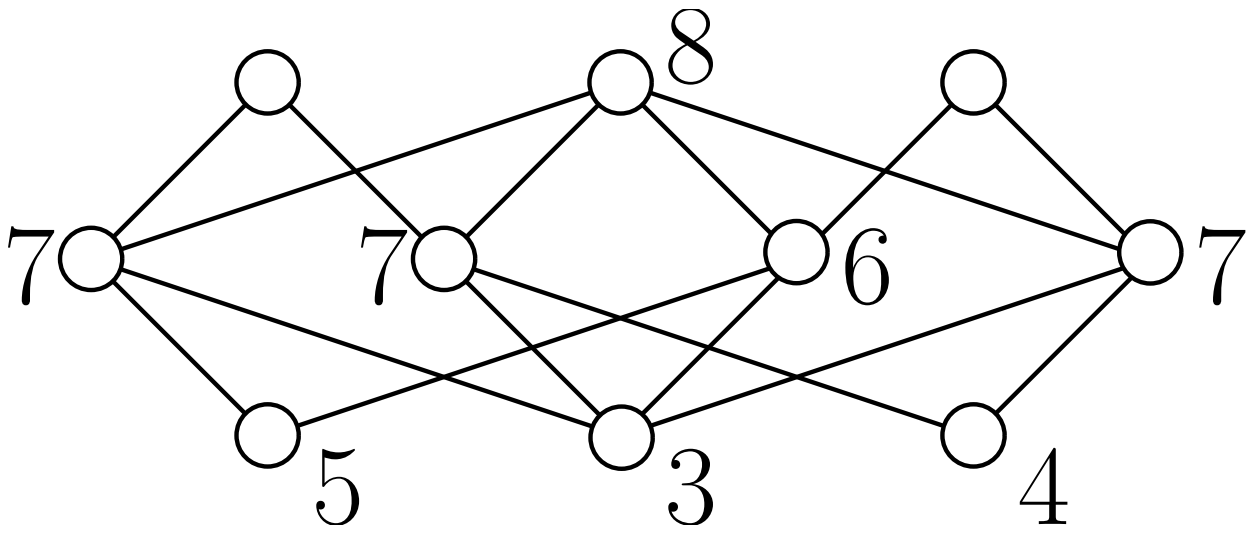
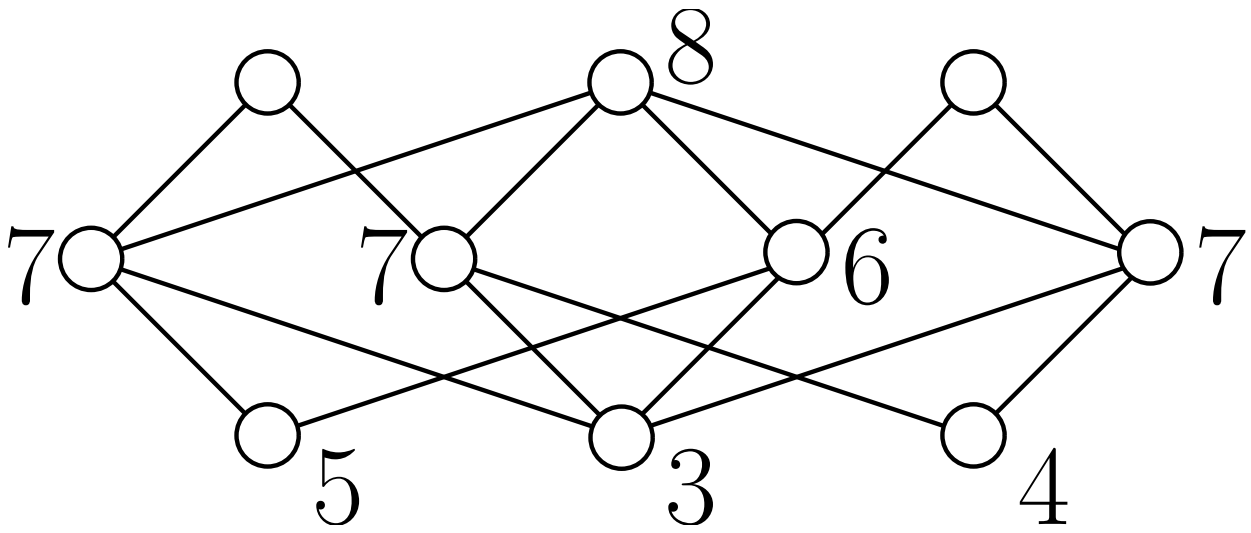


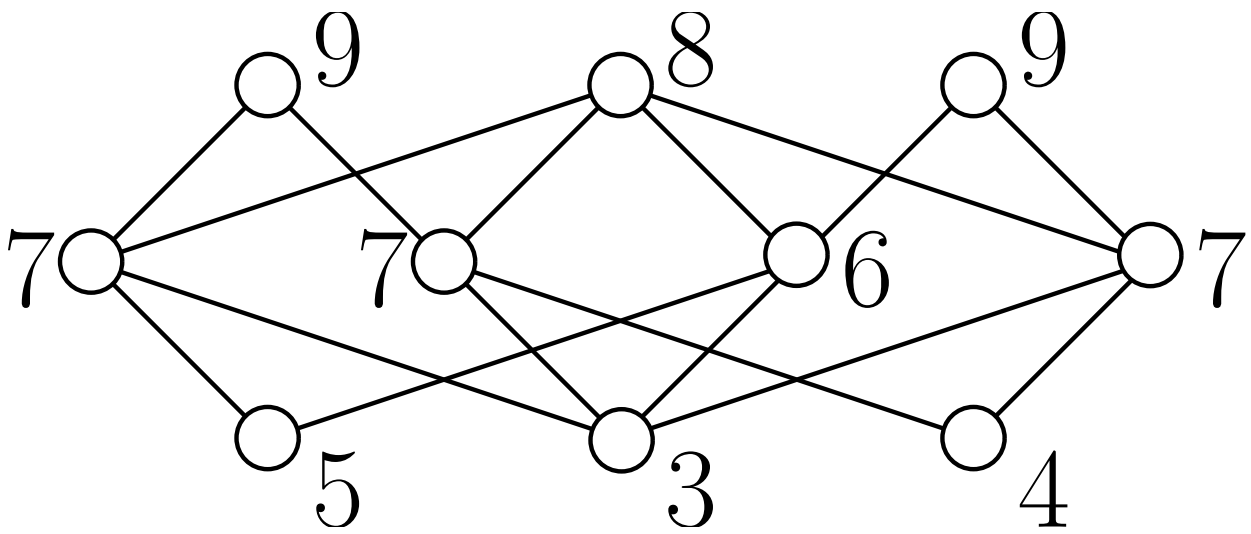
Promotion on increasing labelings $\mathrm{Inc}^q(P)$ by generalized jeu de taquin slides

Consider blank spots and 8's. Fill any blank spot that has a 8 covering it by a 8. Simultaneously, replace any 8's covering blank spots by a blank spot.



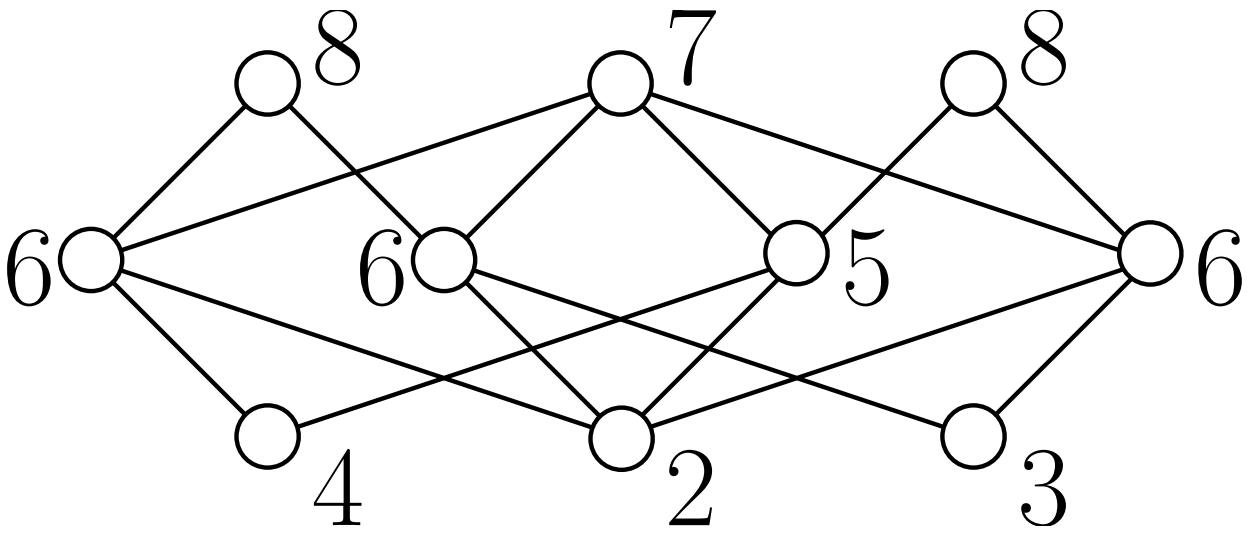
Promotion on increasing labelings $\mathrm{Inc}^q(P)$ by generalized jeu de taquin slides

When all blank spots are maximal, fill the spots with $q+1$.

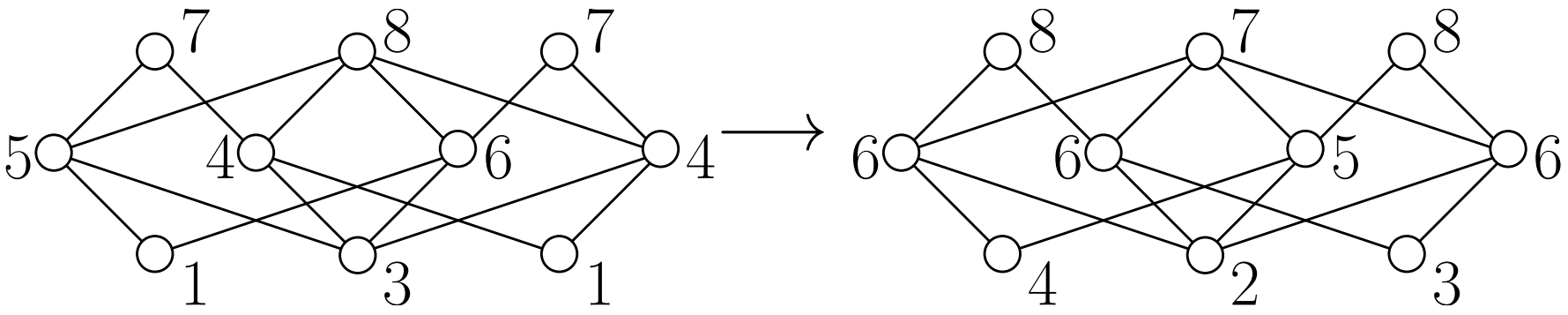


Promotion on increasing labelings $\mathrm{Inc}^q(P)$ by generalized jeu de taquin slides

Subtract 1 from all labels.



Bender-Knuth promotion = jeu de taquin promotion



Theorem (K. Dilks, S., C. Vorland 2018+)

The two definitions of promotion on increasing labelings $\mathrm{Inc}^q(P)$ *coincide.*

Corollary (K. Dilks, S., C. Vorland 2018+)

$(\mathrm{Inc}^P(q), \langle \mathsf{IncPro} \rangle, \mathrm{Con})$ *exhibits resonance with frequency* q.

Goal

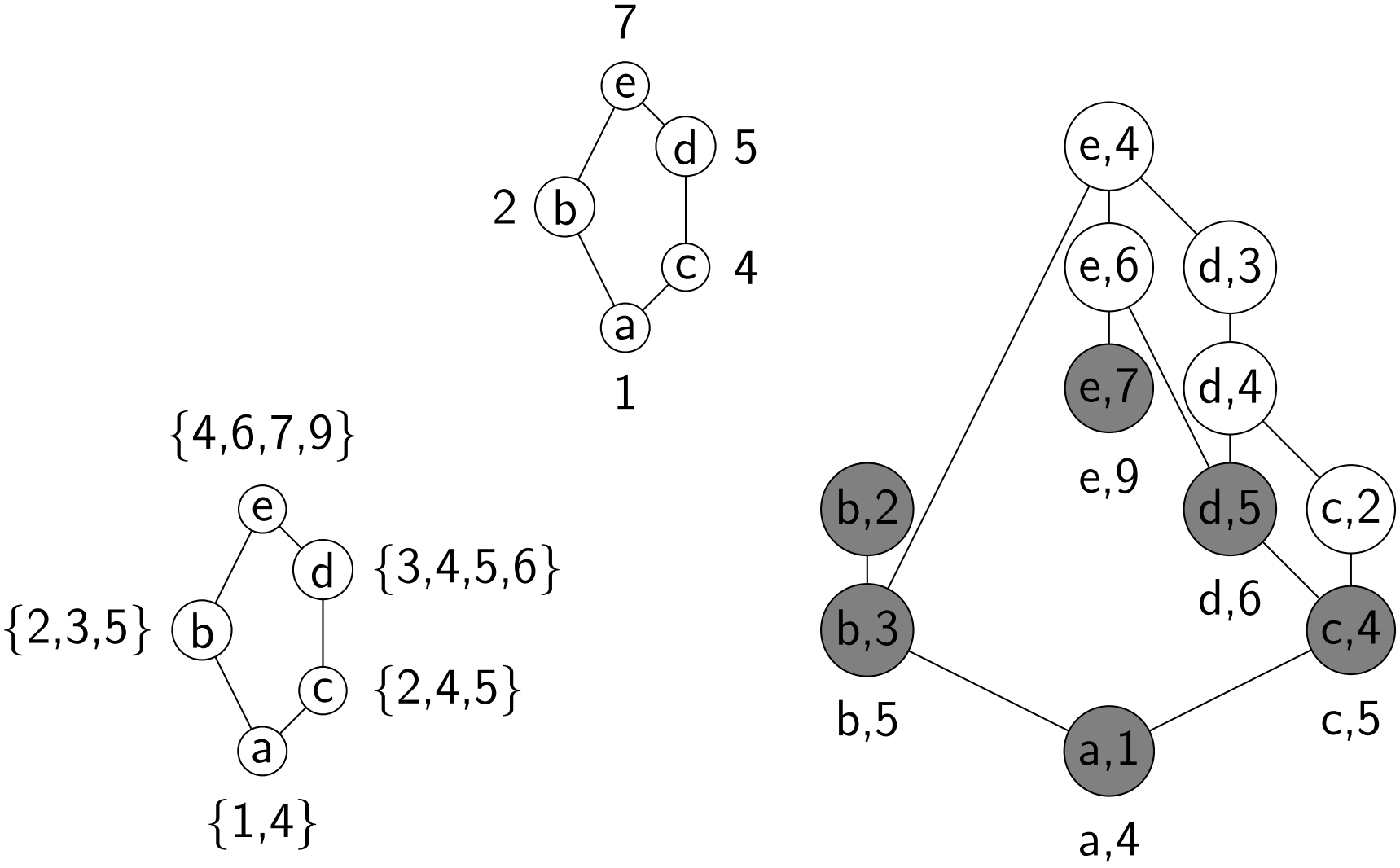
Relate increasing labeling promotion to rowmotion on the order ideals of some poset.

Idea:

- $\mathrm{Inc}^R(P)$ can be partially ordered by componentwise comparison.
- $\mathrm{Inc}^R(P)$ is a distributive lattice where the meet and join are given by taking componentwise min and max.
- Use the Fundamental Theorem of Finite Distributive Lattices.
- Determine how to construct $\Gamma(P, R)$, the subposet of join irreducibles.

Theorem (K. Dilks, S., C. Vorland 2018+)

Order ideals of $\Gamma(P, R)$ *are in bijection with the increasing labelings* $\mathrm{Inc}^R(P)$.



An equivariant bijection between $\mathrm{Inc}^R(P)$ and $J(\Gamma(P,R))$

Definition

$H : P \to \mathbb{Z}$ is a *toggle order* if $p_1 \lessdot p_2$ implies $H(p_1) \neq H(p_2)$. Given a toggle order H, define T_H^i to be the toggle group action that is the product of all t_p for $p \in P$ such that $H(p) = i$.

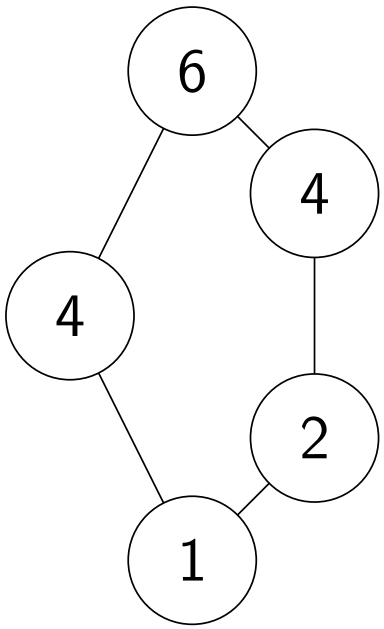
Definition

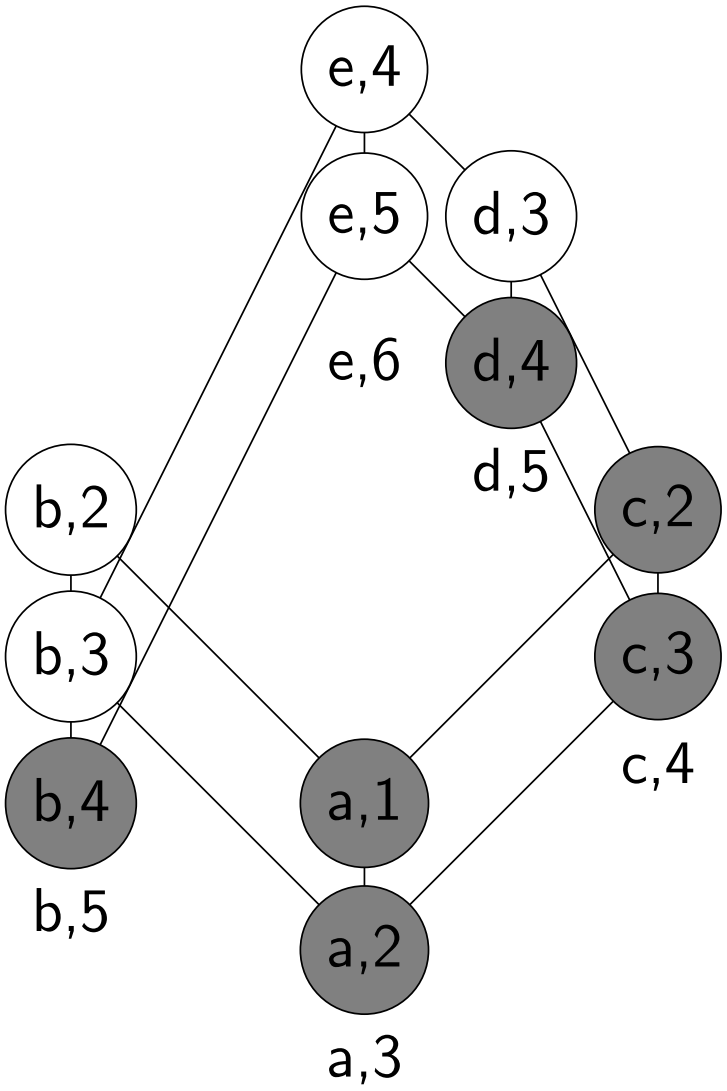
Toggle-promotion with respect to a toggle order H, denoted TogPro_H, is $\ldots T_H^{-2} T_H^{-1} T_H^0 T_H^1 T_H^2 \ldots$

Theorem (K. Dilks, S., C. Vorland 2018+)

Let $H_\Gamma : \Gamma(P,R) \to \mathbb{Z}$ be the toggle order taking (p,k) to k. Then $\mathrm{Inc}^R(P)$ under IncPro *is in equivariant bijection with order ideals of $\Gamma(P,R)$ under* $\mathrm{TogPro}_{H_\Gamma}$.

Each ρ_i involution corresponds to a generalized toggle T_H^i





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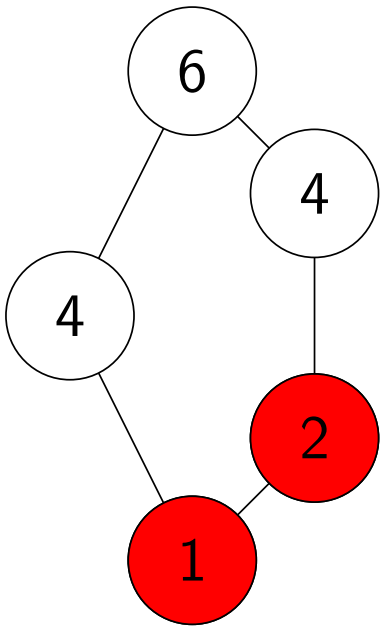

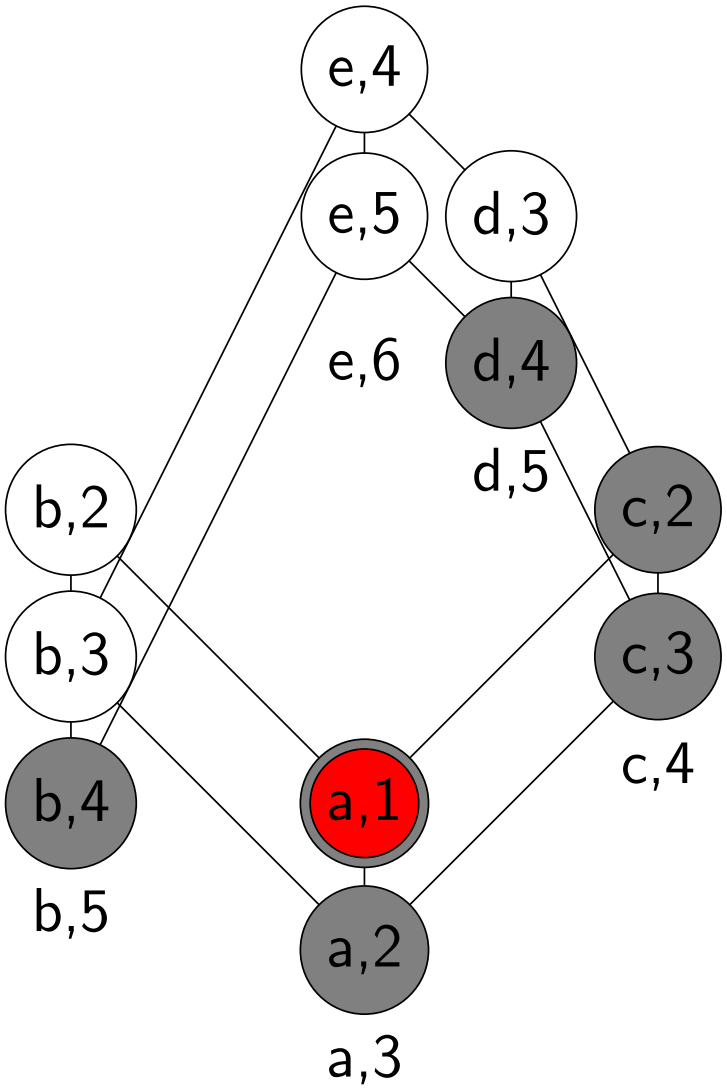

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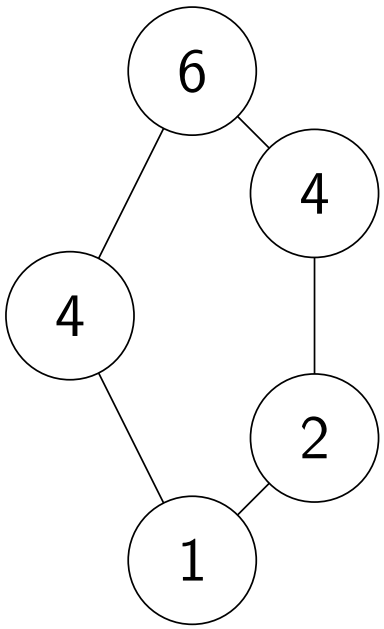

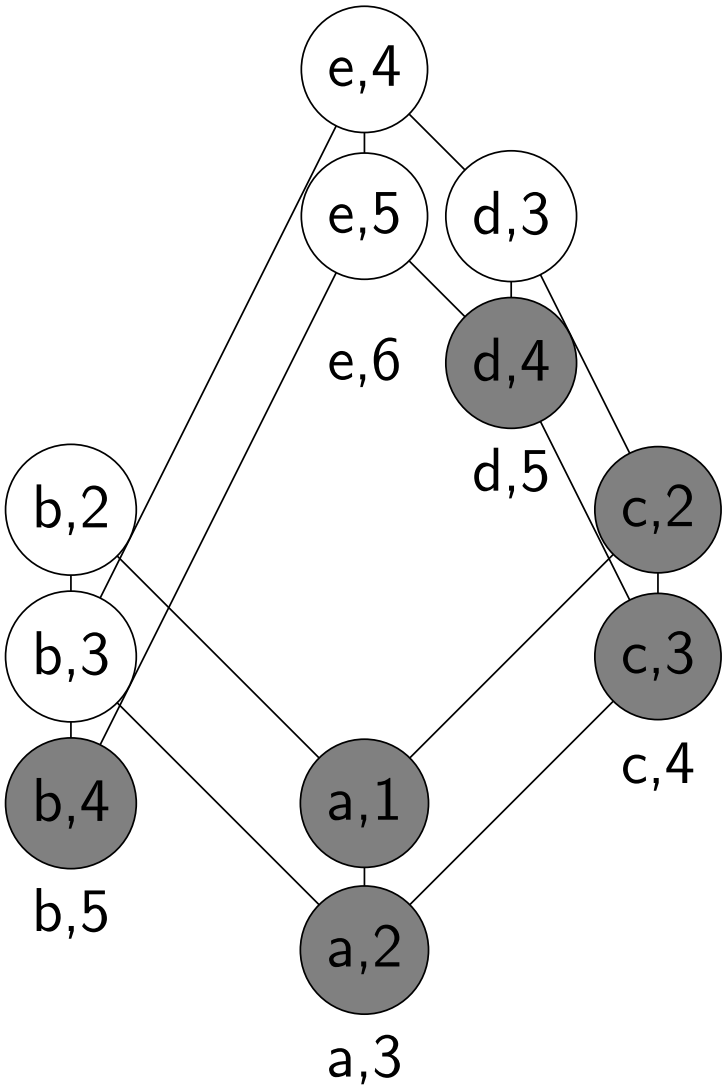

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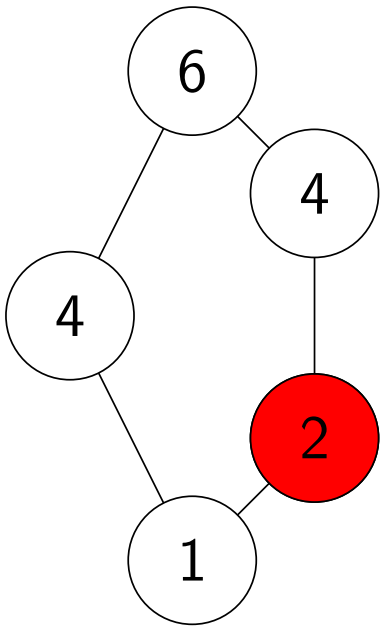

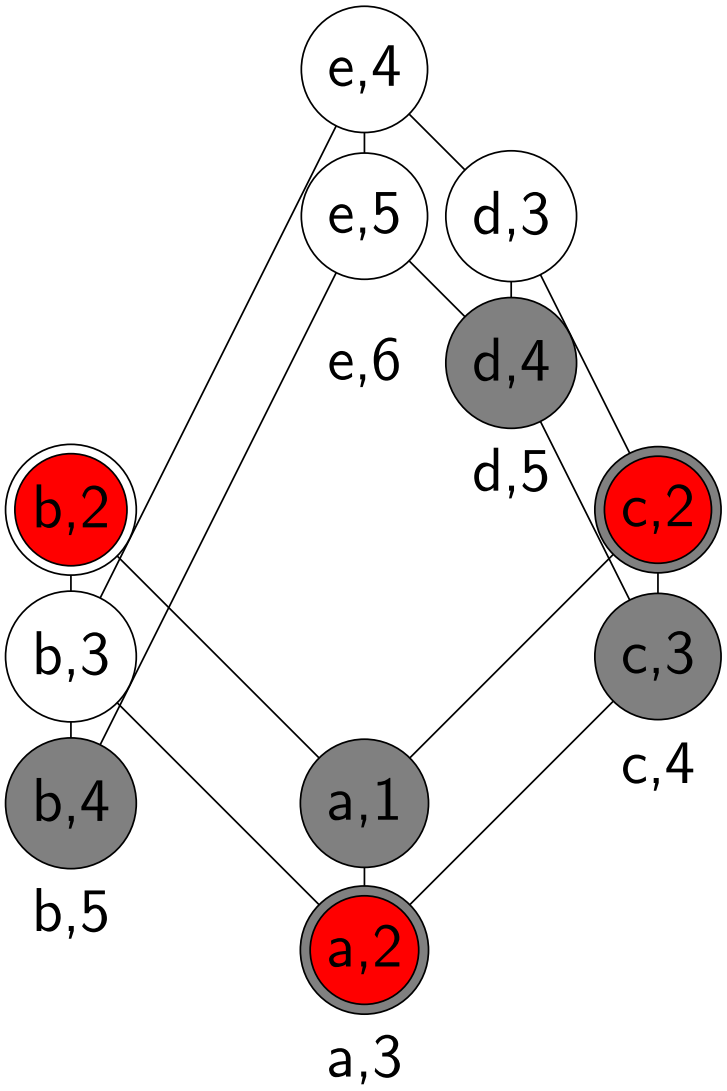

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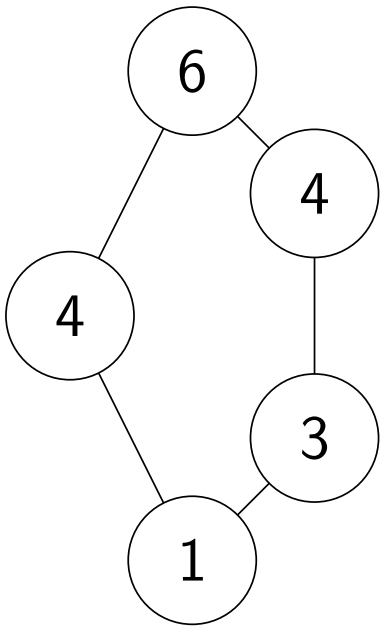

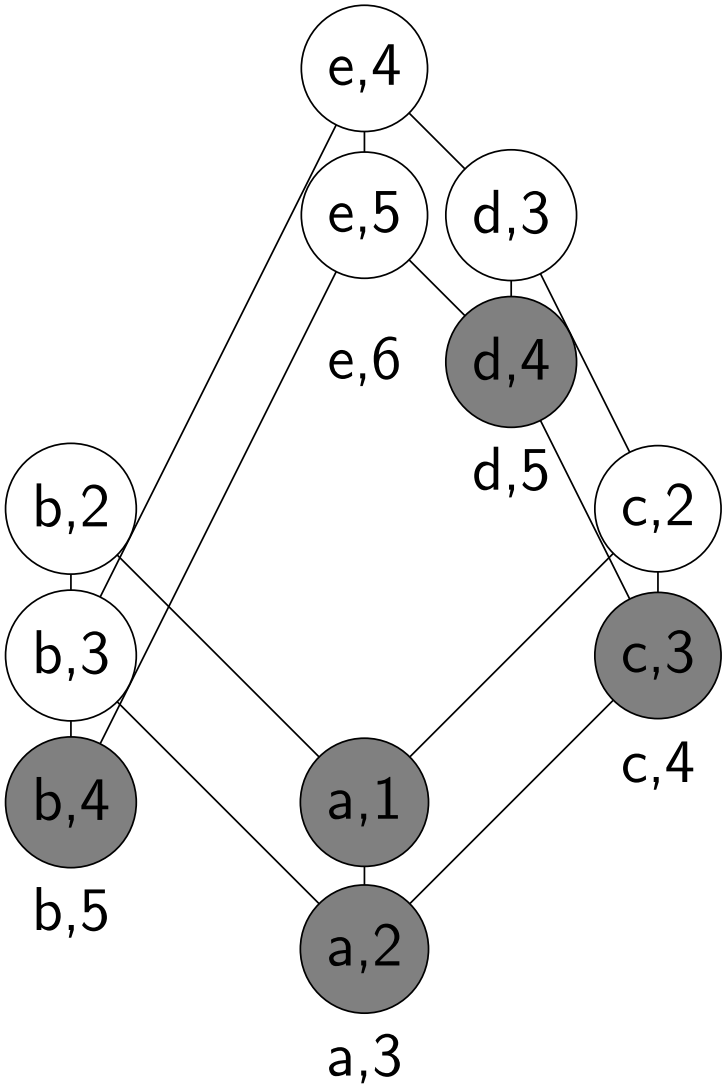

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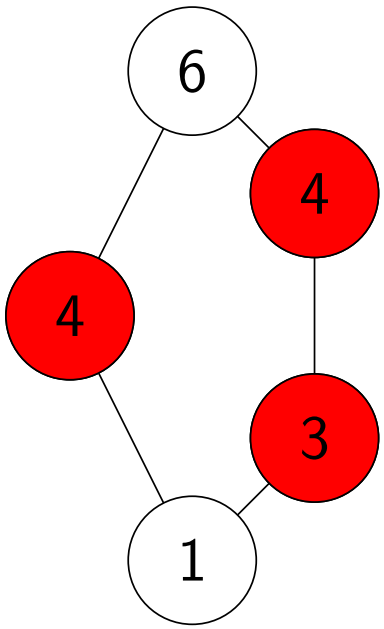

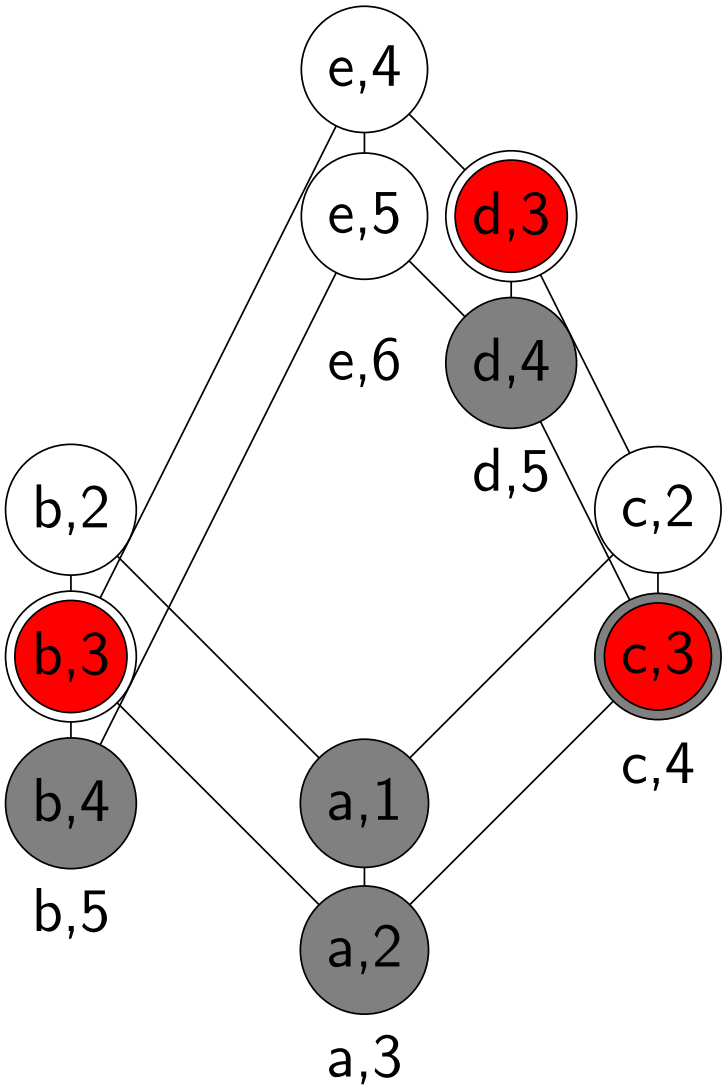

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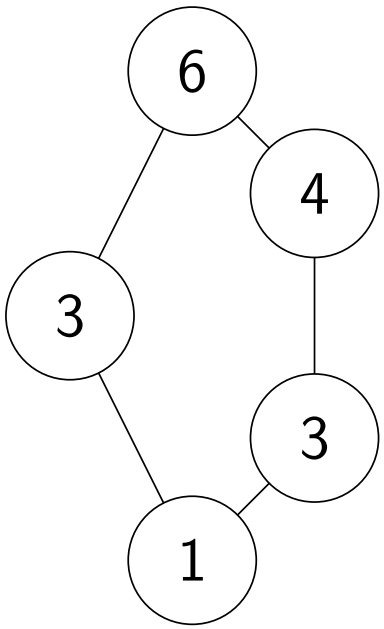

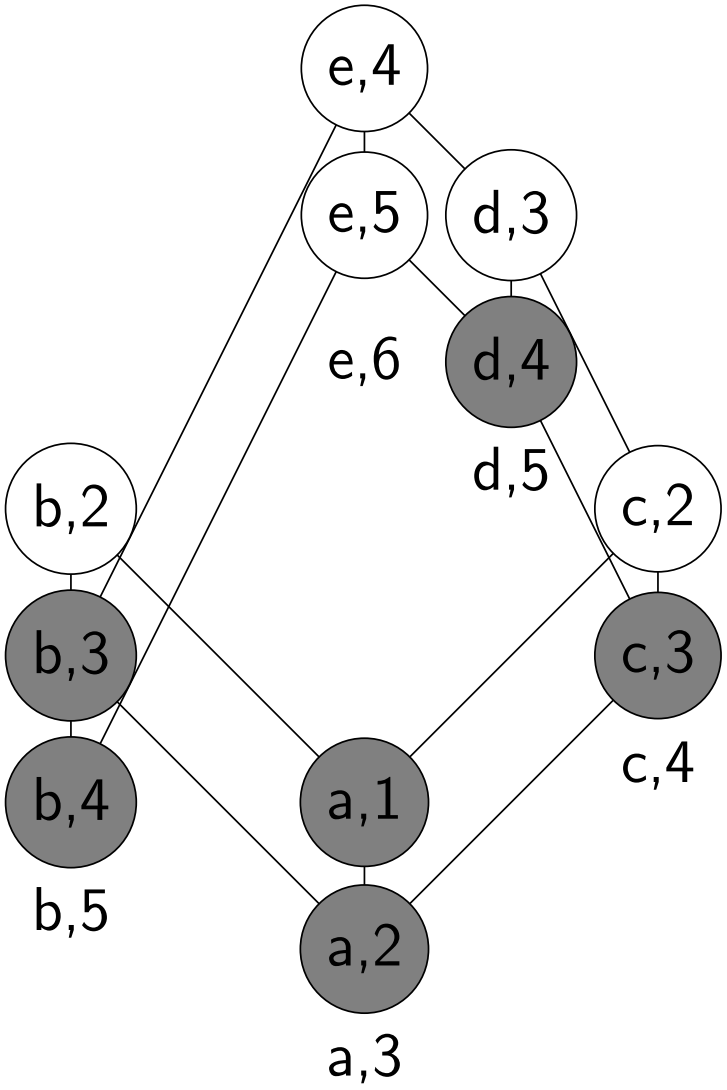

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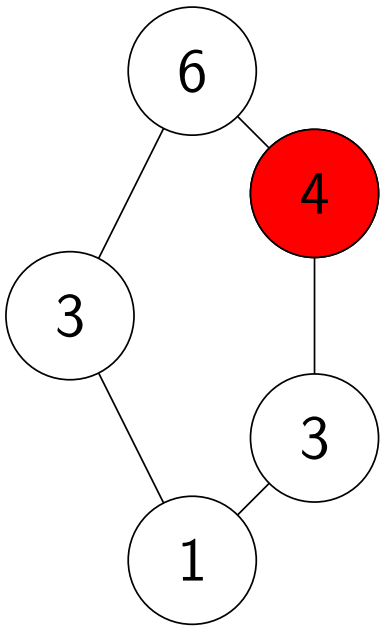

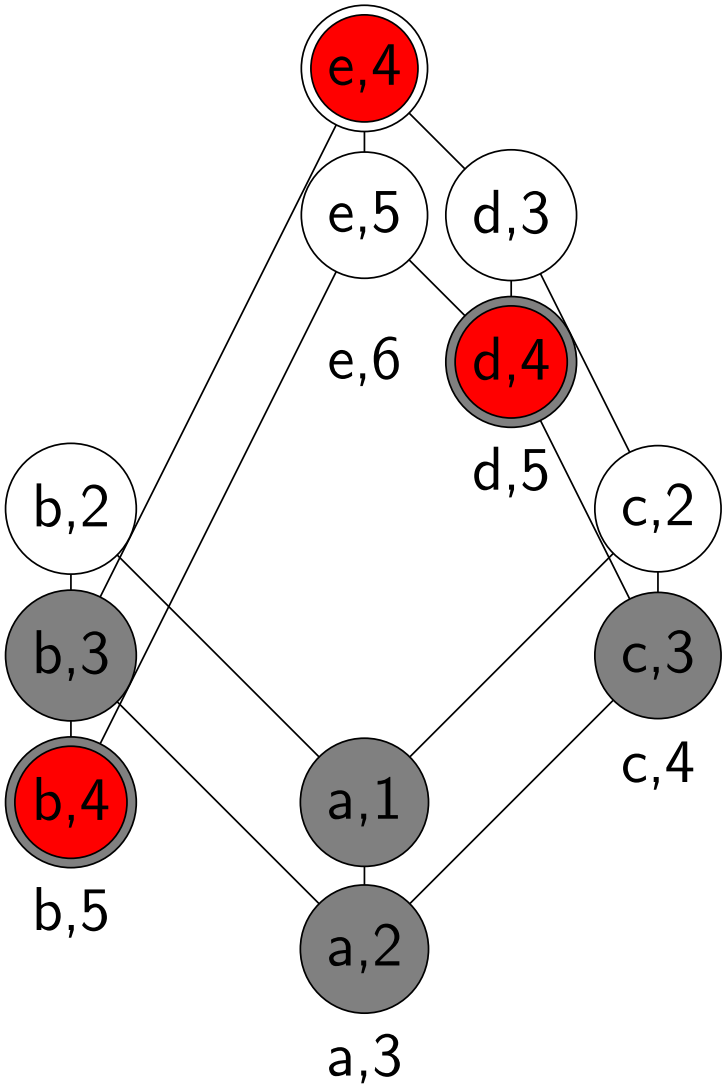

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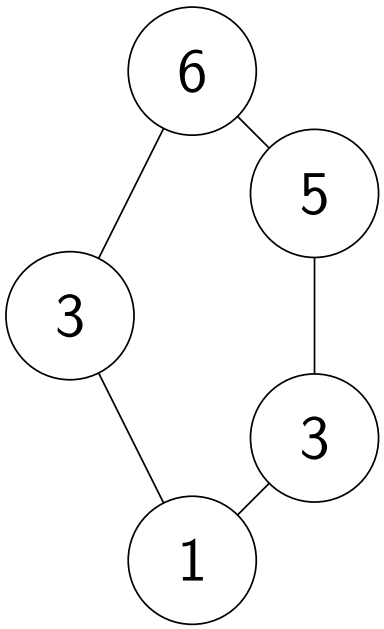

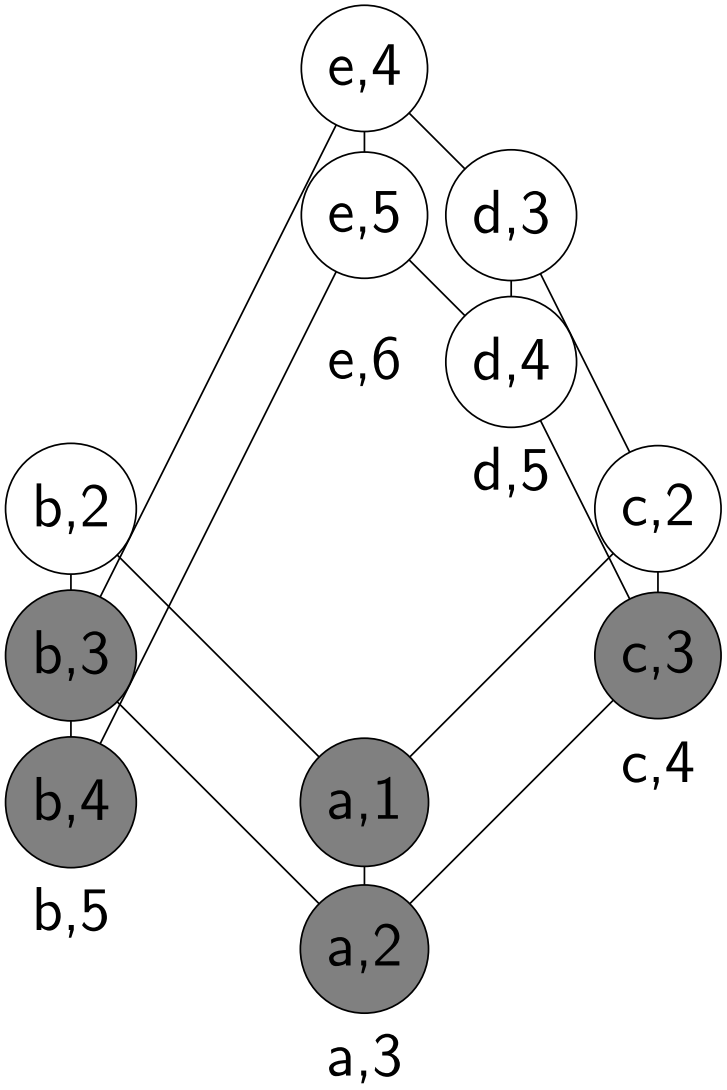

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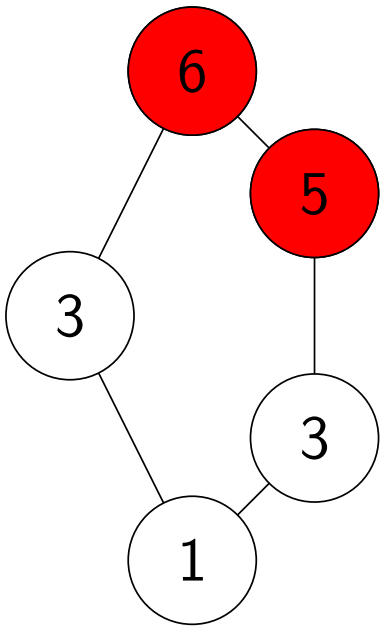

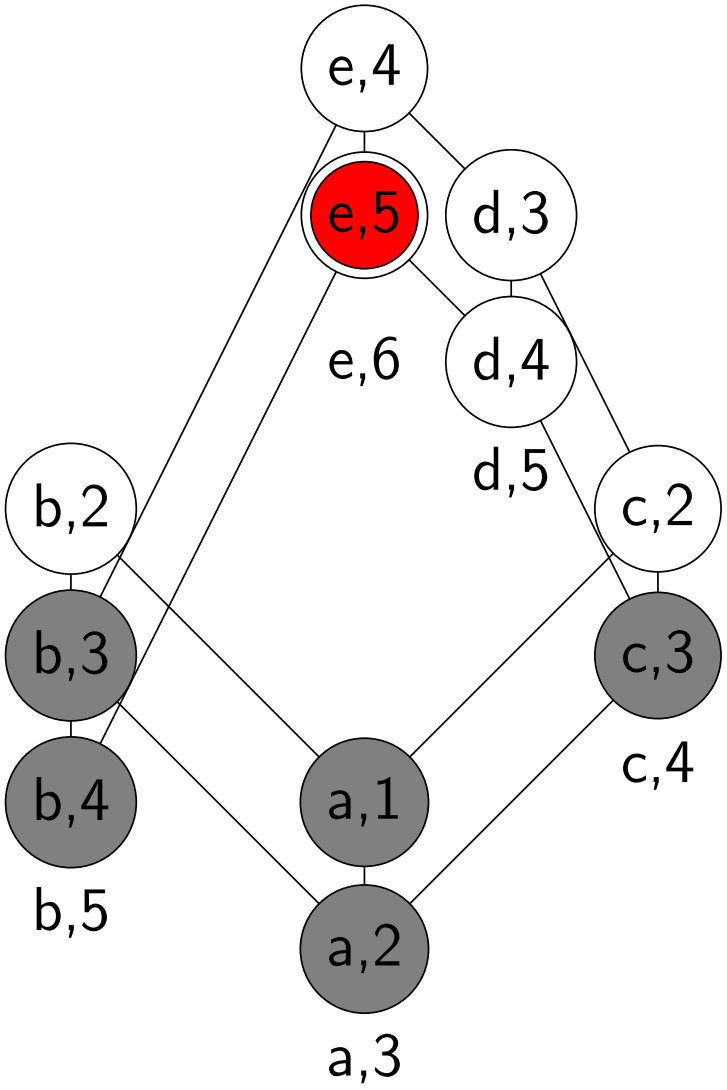

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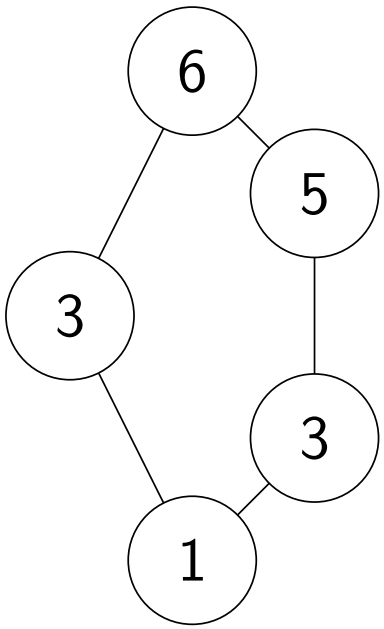
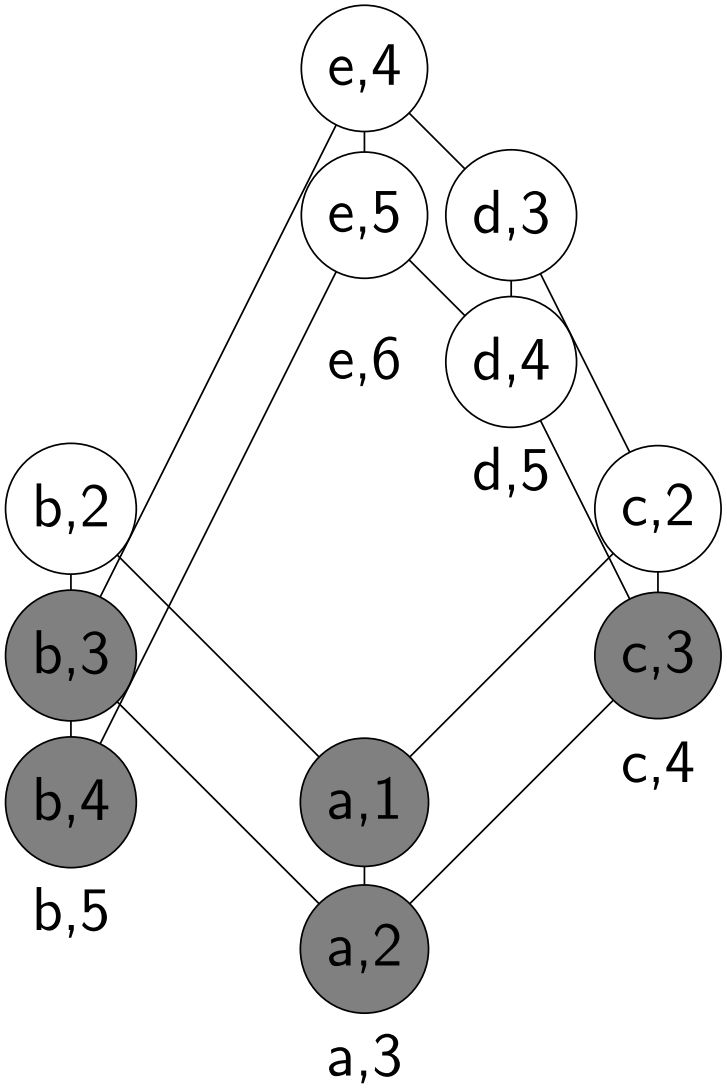




How does TogPro relate to rowmotion?

Definition

$H : P \to \mathbb{Z}$ is a *column toggle order* if whenever $p_1 \lessdot p_2$ in P, then $H(p_1) = H(p_2) \pm 1$.

Theorem (K. Dilks, S., C. Vorland 2018+)

When H_Γ is a column toggle order, there is an equivariant bijection between $\mathrm{Inc}^R(P)$ under IncPro *and $J(\Gamma(P, R))$ under Row.*

Corollary

There is an equivariant bijection between $\mathrm{Inc}^q(P)$ under IncPro *and order ideals in $\Gamma(P, q)$ under Row.*

How does TogPro relate to rowmotion?

Corollary

Let P be a ranked poset with a Cartesian embedding into ranked posets (P_1, P_2). Let H map the element of P embedded at coordinate (p_1, p_2) to the difference of ranks $\mathrm{rk}_{P_1}(p_1) - \mathrm{rk}_{P_2}(p_2)$. Then there is an equivariant bijection on $J(P_1 \times P_2)$ between TogPro_H *and* Row.

Hyperplane promotion $\mathrm{Pro}_{\pi,v}$ with respect to a lattice embedding π can be thought of a special case of this.

Conclusion

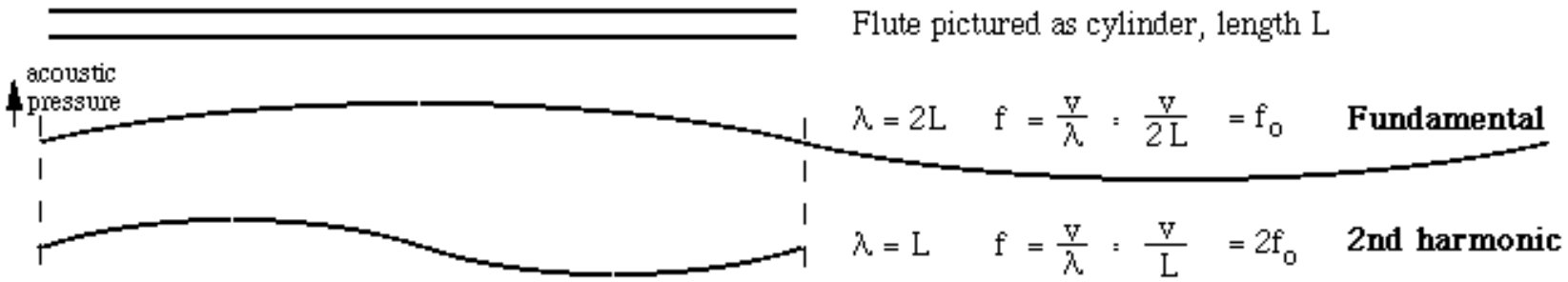
We have seen examples of higher-dimensional combinatorial objects with natural actions that no longer have a nice order, but rather exhibit resonance.

Question

Can we find an analogues of cyclic sieving and homomesy for combinatorial actions that exhibit resonance?

In tomorrow's talk, we will see another example of toggling and resonance in the setting of alternating sign matrices, along with some additional open questions.

Dynamical algebraic combinatorics: Resonance



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- J. Striker, Dynamical algebraic combinatorics: promotion, rowmotion, and resonance. *Notices of the AMS*, **64** (2017), no. 6, 543–549.
- J. Striker and N. Williams, Promotion and rowmotion, *Eur. J. Combin.* **33** (2012), no. 8, 1919–1942.
- K. Dilks, O. Pechenik, and J. Striker, Resonance in orbits of plane partitions and increasing tableaux. *J. Combin. Series A*, **148** (2017) 244–274.
- K. Dilks, J. Striker, and C. Vorland, Rowmotion and increasing labeling promotion. *Submitted*, `arXiv:1710.07179`.