# ORTHOGONAL POLYNOMIAL CONTRASTS

## INDIVIDUAL DF COMPARISONS: EQUALLY SPACED TREATMENTS

- Many treatments are equally spaced (incremented).

- This provides us with the opportunity to look at the response curve of the data (form of multiple regression).

- Regression analysis could be performed using the data; however, when there are equal increments between successive levels of a factor, a simple coding procedure may be used to accomplish the analysis with much less effort.

- This technique is related to the one we will use for linear contrasts.

- We will partition the factor in the ANOVA table into separate single degree of freedom comparisons.

- The number of possible comparisons is equal to the number of levels of a factor minus one.

- For example, if there are three levels of a factor, there are two possible comparisons.

- The comparisons are called orthogonal polynomial contrasts or comparisons.

- Orthogonal polynomials are equations such that each is associated with a power of the independent variable (e.g. X, linear; $X^2$, quadratic; $X^3$, cubic, etc.).

  - $1^{st}$ order comparisons measure linear relationships.
  - $2^{nd}$ order comparisons measures quadratic relationships.
  - $3^{rd}$ order comparisons measures cubic relationships.

<u>Example 1</u>

- We wish to determine the effect of drying temperature on germination of barley, we can set up an experiment that uses three equally space temperatures (90$^o$F, 100$^o$F, 110$^o$F) and four replicates.

- The ANOVA would look like:

| Sources of variation | df |
|---|---|
| Rep | r-1=3 |
| Treatment | t-1=2 |
| Error | (r-1)(t-1)=6 |
| Total | rt-1=11 |

- Since there are three treatments equally spaced by 10°F, treatment can be partitioned into two single df polynomial contrasts (i.e. linear and quadratic).

- The new ANOVA table would look like:

| Sources of variation | df | |
|---|---|---|
| Replicate | r-1 | |
| Treatment | t-1 | |
| Linear | | 1 |
| Quadratic | | 1 |
| Error | (r-1)(t-1) | |
| Total | rt-1 | |

- Sum of squares can be calculated for each of the comparisons using the formula: $\mathbf{Q^2 / (k*r)}$ where $Q=\Sigma c_i Y_{i.}$, $k=\Sigma c_i^2$, c=coefficients, and r=number of replicates.

- *F*-tests can be calculated for each of the polynomial contrasts.

The coefficients used for calculating sums of squares are:

| | | Treatment totals | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Number of treatment | Degree of polynomial | T1 | T2 | T3 | T4 | T5 | T6 | Divisor $k=\sum c_i^2$ |
| 2 | 1 | -1 | +1 | | | | | 2 |
| 3 | 1 | -1 | 0 | +1 | | | | 2 |
| | 2 | +1 | -2 | +1 | | | | 6 |
| 4 | 1 | -3 | -1 | +1 | +3 | | | 20 |
| | 2 | +1 | -1 | -1 | +1 | | | 4 |
| | 3 | -1 | +3 | -3 | +1 | | | 20 |
| 5 | 1 | -2 | -1 | 0 | +1 | +2 | | 10 |
| | 2 | +2 | -1 | -2 | -1 | +2 | | 14 |
| | 3 | -1 | +2 | 0 | -2 | +1 | | 10 |
| | 4 | +1 | -4 | +6 | -4 | +1 | | 70 |
| 6 | 1 | -5 | -3 | -1 | +1 | +3 | +5 | 70 |
| | 2 | +5 | -1 | -4 | -4 | -1 | +5 | 84 |
| | 3 | -5 | +7 | +4 | -4 | -7 | +5 | 180 |
| | 4 | +1 | -3 | +2 | +2 | -3 | +1 | 28 |
| | 5 | -1 | +5 | -10 | +10 | -5 | +1 | 252 |

.

- Coefficients for these comparisons would be as follows:

| | Drying temperature | | |
|---|---|---|---|
| | $90^o$ | $100^o$ | $110^o$ |
| Linear | -1 | 0 | 1 |
| Quadratic | 1 | -2 | 1 |

Using the formulas $\mathbf{SS = Q^2/ (k*r)}$ $Q= \Sigma c_i Y_{i.}$ and $k=\mathbf{\Sigma c_i^2}$

$SS_{Linear} = [(-1)(Y_{1.)} + (0)(Y_{2.}) – (1)(Y_{3.})]^2 \div (2*4)$

$SS_{Quadradtic} = [(1)(Y_{1.)} + (-2)(Y_{2.}) – (1)(Y_{3.})]^2 \div (6*4)$

Example 2

Effect of row spacing on yield (bu/ac) of soybean.

| | Row spacing (inches) | | | | | |
|---|---|---|---|---|---|---|
| Block | 18 | 24 | 30 | 36 | 42 | $\Sigma Y_{.j}$ |
| 1 | 33.6 | 31.1 | 33.0 | 28.4 | 31.4 | 157.5 |
| 2 | 37.1 | 34.5 | 29.5 | 29.9 | 28.3 | 159.3 |
| 3 | 34.1 | 30.5 | 29.2 | 31.6 | 28.9 | 154.3 |
| 4 | 34.6 | 32.7 | 30.7 | 32.3 | 28.6 | 158.9 |
| 5 | 35.4 | 30.7 | 30.7 | 28.1 | 29.6 | 154.5 |
| 6 | 36.1 | 30.3 | 27.9 | 26.9 | 33.4 | 154.6 |
| $Y_{i.}$ | 210.9 | 189.8 | 181.0 | 177.2 | 180.2 | 939.1 |
| $\overline{Y}_{i.}$ | 35.15 | 31.63 | 30.17 | 29.53 | 30.03 | 31.3 |

Step 1. ANOVA of data as an RCBD:

| SOV | Df | SS | MS | F |
|---|---|---|---|---|
| Rep | 5 | 5.41 | | |
| Treatment | 4 | 125.66 | 31.415 | $8.4997^{**}$ |
| Error | 20 | 73.92 | 3.696 | |
| Total | 29 | 204.99 | | |

Step 2. Partition Treatment of source of variation into four single degree of freedom orthogonal polynomial contrasts.

| | Row spacing (inches) | | | | |
|---|---|---|---|---|---|
| Contrast | 18 | 24 | 30 | 36 | 42 |
| $\Sigma Y_i$ | 210.9 | 189.8 | 181.0 | 177.2 | 180.0 |
| Linear | -2 | -1 | 0 | 1 | 2 |
| Quadratic | 2 | -1 | -2 | -1 | 2 |
| Cubic | -1 | 2 | 0 | -2 | 1 |
| Quartic | 1 | -4 | 6 | -4 | 1 |

Step 3. Calculate Sum of Squares for each contrast.

| | Row spacing (inches) | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Contrast | 18 | 24 | 30 | 36 | 42 | | | |
| $\Sigma Y_{i.}$ | 210.9 | 189.8 | 181.0 | 177.2 | 180.0 | Q | $r\sum c_i^2$ | SS |
| Linear | -2 | -1 | 0 | 1 | 2 | -74.4 | 6(10) | 92.25 |
| Quadratic | 2 | -1 | -2 | -1 | 2 | 52.8 | 6(14) | 33.19 |
| Cubic | -1 | 2 | 0 | -2 | 1 | -5.7 | 6(10) | 0.54 |
| Quartic | 1 | -4 | 6 | -4 | 1 | 8.9 | 6(70) | 0.19 |

Remember:

$Q=\Sigma c_i Y_{i.}$

e.g. Q for Linear = (-2)210.9 + (-1)189.8 + (0)181.0 + (1)177.2 + (2)180.0 = -74.4

and SS = $Q^2/rk$

Step 4. Rewrite ANOVA.

| SOV | Df | SS | MS | F |
|---|---|---|---|---|
| Rep | 5 | 5.41 | | |
| Treatment | 4 | 126.17 | 31.54 | TRT MS/Error MS = 8.59$^{**}$ |
| Linear | 1 | 92.25 | 92.25 | Lin. MS/Error MS = 24.14$^{**}$ |
| Quadratic | 1 | 33.19 | 33.19 | Quad. MS/Error MS = 9.04$^{**}$ |
| Cubic | 1 | 0.54 | 0.54 | Cubic MS/Error MS = 0.15 |
| Quartic | 1 | 0.19 | 0.19 | Quart. MS/Error MS = 0.05 |
| Error | 20 | 73.41 | 3.67 | |
| Total | 29 | 204.99 | | |

Step 5. Conclusions

This analysis shows highly significant linear and quadratic effects for the row spacing treatments.

The linear component is the portion of the SS attributable to the linear regression of yield on spacing.

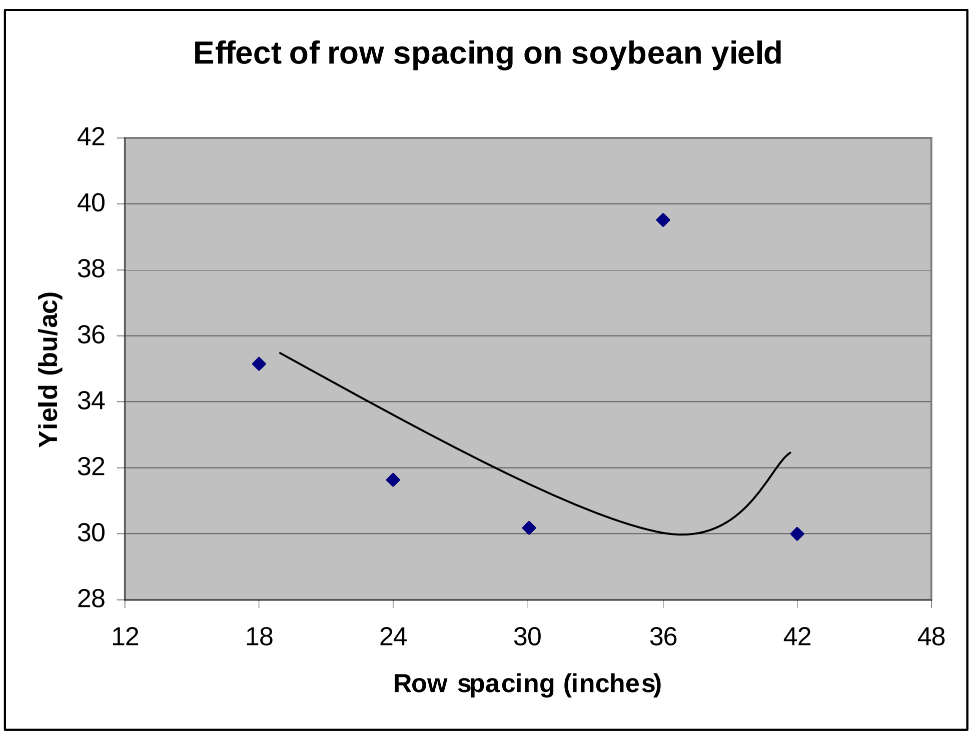
$$\text{i.e. } b_1 = \frac{SSCP}{r(\text{SS X})} = \frac{\sum XY - \frac{\sum X \sum Y}{n}}{r(\text{SS X})} = \frac{\sum XY}{r(\text{SS X})} = \frac{Q}{r\sum c_i^2}$$

Therefore $b_1 = -74.4/60 = -1.24$

The quadratic component measures the additional improvement due to fitting the $X^2$ component.

$$\text{i.e. } b_2 = \frac{SSCP}{r(\text{SS X})} = \frac{\sum XY - \frac{\sum X \sum Y}{n}}{r(\text{SS X})} = \frac{\sum XY}{r(\text{SS X})} = \frac{Q}{r\sum c_i^2}$$

Therefore $b_2 = 52.8/(6*14) = 0.63$.



Example 3

Given the following data set, partition the A and B main effects, and the AxB interaction using orthogonal polynomial contrasts.

| Treatment | Rep 1 | Rep 2 | Rep 3 | Trt Total |
|---|---|---|---|---|
| $a_0b_0$ | 10 | 12 | 14 | 36 |
| $a_0b_1$ | 16 | 19 | 21 | 56 |
| $a_0b_2$ | 24 | 27 | 32 | 82 |
| $a_1b_0$ | 12 | 15 | 17 | 44 |
| $a_1b_1$ | 19 | 23 | 29 | 71 |
| $a_1b_2$ | 33 | 34 | 37 | 104 |
| $a_2b_0$ | 24 | 27 | 29 | 80 |
| $a_2b_1$ | 39 | 41 | 43 | 123 |
| $a_2b_2$ | 45 | 47 | 52 | 144 |
| Rep Total | 222 | 245 | 273 | 740 |

Step 1. Complete the ANOVA as an RCBD.

| SOV | df | SS | MS | F |
|---|---|---|---|---|
| Rep | 2 | 144.9 | 72.5 | 60.4** |
| A | 2 | 1790.3 | 895.2 | 746.0** |
| B | 2 | 1607.4 | 803.7 | 669.8** |
| AxB | 4 | 58.8 | 14.7 | 7.4** |
| Error | 16 | 19.1 | 1.2 | |
| Total | 26 | 3620.5 | | |

Step 2. Partition A, B, and AxB into their polynomial components.

| Contrast | $a_0b_0$ | $a_0b_1$ | $a_0b_2$ | $a_1b_0$ | $a_1b_1$ | $a_1b_2$ | $a_2b_0$ | $a_2b_1$ | $a_2b_2$ | Q | k |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $A_L$ | -1 | -1 | -1 | 0 | 0 | 0 | 1 | 1 | 1 | 173 | 6 |
| $A_Q$ | 1 | 1 | 1 | -2 | -2 | -2 | 1 | 1 | 1 | 83 | 18 |
| $B_L$ | -1 | 0 | 1 | -1 | 0 | 1 | -1 | 0 | 1 | 170 | 6 |
| $B_Q$ | 1 | -2 | 1 | 1 | -2 | 1 | 1 | -2 | 1 | -10 | 18 |
| $A_LB_L$ | 1 | 0 | -1 | 0 | 0 | 0 | -1 | 0 | 1 | 18 | 4 |
| $A_LB_Q$ | -1 | 2 | -1 | 0 | 0 | 0 | 1 | -2 | 1 | -28 | 12 |
| $A_QB_L$ | -1 | 0 | 1 | 2 | 0 | -2 | -1 | 0 | 1 | -10 | 12 |
| $A_QB_Q$ | 1 | -2 | 1 | -2 | 4 | -2 | 1 | -2 | 1 | -28 | 36 |

Step 3. Calculate SS for each contrast.

| Contrast | $Q^2/rk$ | SS |
|---|---|---|
| SS $A_L$ | $173^2/(3x6)$ = | 1662.7 |
| SS $A_Q$ | $83^2/(3x18)$ = | 127.6 |
| SS $B_L$ | $170^2/(3x6)$ = | 1605.6 |
| SS $B_Q$ | $-10^2/(3x18)$= | 1.9 |
| SS $A_LB_L$ | $18^2/(3x4)$= | 27.0 |
| SS $A_LB_Q$ | $-28^2/(3x12)$= | 21.8 |
| SS $A_QB_L$ | $-10^2/(3x12)$ = | 2.8 |
| SS $A_QB_Q$ | $-28^2/(3x36)$ = | 7.3 |

Step 4. Rewrite the ANOVA.

| SOV | df | SS | MS | F (assuming A and B fixed) | |
|---|---|---|---|---|---|
| Rep | 2 | 144.9 | 72.5 | 60.4** | |
| A | 2 | 1790.3 | 895.2 | 746.0** | |
| $A_L$ | 1 | 1662.7 | 1662.7 | | 1385.6** |
| $A_Q$ | 1 | 127.6 | 127.6 | | 106.3** |
| B | 2 | 1607.4 | 803.7 | 669.8** | |
| $B_L$ | 1 | 1605.6 | 1605.6 | | 1338.0** |
| $B_Q$ | 1 | 1.9 | 1.9 | | 1.6 |
| AxB | 4 | 58.8 | 14.7 | 7.4** | |
| $A_LB_L$ | 1 | 27.0 | 27.0 | | 22.5** |
| $A_LB_Q$ | 1 | 21.8 | 21.8 | | 18.2** |
| $A_QB_L$ | 1 | 2.8 | 2.8 | | 2.3 |
| $A_QB_Q$ | 1 | 7.3 | 7.3 | | 6.1* |
| Error | 16 | 19.1 | 1.2 | | |
| Total | 26 | 3620.5 | | | |

Step 5. Conclusions

The $A_L$, $B_L$, $A_LxB_L$, $A_LxB_Q$, and $A_QB_Q$ sources of variation all contribute significantly to the model.

**Analysis of Orthogonal Polynomial Contrasts**

SAS Commands

```
options pageno=1;
data orthpoly;
input row rep yield;
datalines;
18     1      33.6
24     1      31.1
30     1      33
36     1      28.4
42     1      31.4
18     2      37.1
24     2      34.5
30     2      29.5
36     2      29.9
42     2      28.3
18     3      34.1
24     3      30.5
30     3      29.2
36     3      31.6
42     3      28.9
18     4      34.6
24     4      32.7
30     4      30.7
36     4      32.3
42     4      28.6
18     5      35.4
24     5      30.7
30     5      30.7
36     5      28.1
42     5      29.6
18     6      36.1
24     6      30.3
30     6      27.9
36     6      26.9
42     6      33.4
;;
ods rtf file='polynomial contrast SAS output.rtf';
proc print;
title 'Printout of Orthogonal Polynomial Data';
run;
Proc glm;
Class rep row;
Model yield=rep row;
*Treatment order-------------------18     24    30    36    42;
contrast 'linear'       row              -2    -1    0     1     2;
contrast 'quadratic'    row              2     -1    -2    -1    2;
contrast 'cubic'        row              -1    2     0     -2    1;
contrast 'quartic'            row              1     -4    6     -4    1;
title 'Anova with Polynomial Contrasts';
run;
ods rtf close;
run
```

## *Printout of Orthogonal Polynomial Data*

| Obs | row | rep | yield |
|---|---|---|---|
| 1 | 18 | 1 | 33.6 |
| 2 | 24 | 1 | 31.1 |
| 3 | 30 | 1 | 33.0 |
| 4 | 36 | 1 | 28.4 |
| 5 | 42 | 1 | 31.4 |
| 6 | 18 | 2 | 37.1 |
| 7 | 24 | 2 | 34.5 |
| 8 | 30 | 2 | 29.5 |
| 9 | 36 | 2 | 29.9 |
| 10 | 42 | 2 | 28.3 |
| 11 | 18 | 3 | 34.1 |
| 12 | 24 | 3 | 30.5 |
| 13 | 30 | 3 | 29.2 |
| 14 | 36 | 3 | 31.6 |
| 15 | 42 | 3 | 28.9 |
| 16 | 18 | 4 | 34.6 |
| 17 | 24 | 4 | 32.7 |
| 18 | 30 | 4 | 30.7 |
| 19 | 36 | 4 | 32.3 |
| 20 | 42 | 4 | 28.6 |
| 21 | 18 | 5 | 35.4 |
| 22 | 24 | 5 | 30.7 |
| 23 | 30 | 5 | 30.7 |
| 24 | 36 | 5 | 28.1 |
| 25 | 42 | 5 | 29.6 |
| 26 | 18 | 6 | 36.1 |
| 27 | 24 | 6 | 30.3 |
| 28 | 30 | 6 | 27.9 |
| 29 | 36 | 6 | 26.9 |
| 30 | 42 | 6 | 33.4 |

## *Anova with Polynomial Contrasts*

### *The GLM Procedure*

| Class Level Information | | |
|---|---|---|
| **Class** | **Levels** | **Values** |
| **rep** | 6 | 1 2 3 4 5 6 |
| **row** | 5 | 18 24 30 36 42 |

| | |
|---|---|
| **Number of Observations Read** | 30 |
| **Number of Observations Used** | 30 |

## *Anova with Polynomial Contrasts*

### *The GLM Procedure*

| Source | DF | Sum of Squares | Mean Square | F Value | Pr > F |
|---|---|---|---|---|---|
| Model | 9 | 131.0710000 | 14.5634444 | 3.94 | 0.0051 |
| Error | 20 | 73.9186667 | 3.6959333 | | |
| Corrected Total | 29 | 204.9896667 | | | |

| R-Square | Coeff Var | Root MSE | yield Mean |
|---|---|---|---|
| 0.639403 | 6.141458 | 1.922481 | 31.30333 |

| Source | DF | Type I SS | Mean Square | F Value | Pr > F |
|---|---|---|---|---|---|
| rep | 5 | 5.4096667 | 1.0819333 | 0.29 | 0.9113 |
| row | 4 | 125.6613333 | 31.4153333 | 8.50 | 0.0004 |

| Source | DF | Type III SS | Mean Square | F Value | Pr > F |
|---|---|---|---|---|---|
| rep | 5 | 5.4096667 | 1.0819333 | 0.29 | 0.9113 |
| row | 4 | 125.6613333 | 31.4153333 | 8.50 | 0.0004 |

| Contrast | DF | Contrast SS | Mean Square | F Value | Pr > F |
|---|---|---|---|---|---|
| linear | 1 | 91.26666667 | 91.26666667 | 24.69 | <.0001 |
| quadratic | 1 | 33.69333333 | 33.69333333 | 9.12 | 0.0068 |
| cubic | 1 | 0.50416667 | 0.50416667 | 0.14 | 0.7158 |
| quartic | 1 | 0.19716667 | 0.19716667 | 0.05 | 0.8197 |