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THREE-DIMENSIONAL ELLIPTIC CRACK UNDER IMPACT LOADING*

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ABSTRACT The dynamic stress intensity factor of a three-dimensional elliptic crack under impact loading is determined with the finite element method. The computation results can take into account the influence of time and the ratio of the wave speeds on the stress intensity factor. The present method is suitable not only for three-dimensional dynamic crack, but also for three-dimensional dynamic contact.

KEY WORDS dynamic loading, three-dimensional elliptic crack, finite element, dynamic stress intensity factor

I. INTRODUCTION

It is well known that the three-dimensional elliptic crack is one of the significant embedded cracks, whose stress intensity factors for the static case have been extensively investigated[1,2] and the Green-Sneddon solution[1] is a beautiful exact analytic solution that plays a very important role in fracture mechanics[3].

As materials and structures are often under the action of dynamic loading, the dynamic stress intensity factor of three-dimensional elliptic crack should be studied. Because of the complexity of the problem, its analytic solution is up to now not available. One of the possible ways to construct the analytic solution of the problems is to reduce it to a two-dimensional dual integral equations. The numerical determination of the dynamic stress intensity factor is available, but the relevant results of which are seldom, if ever, reported. The present paper is intended to study the numerical computation by dynamic finite element and to determine the stress intensity factor. The numerical results cannot be compared with analytic solution (for lack of the analytic solution). A comparison of our results with those of the static Green-Sneddon solution corresponding to the case of $t\to\infty$ and those of the dynamic solution of penny-shaped crack corresponding to the case of $b/a\to 1$ shows that the numerical solution is highly accurate.

II. FINITE ELEMENT FORMULATION

For the dynamic elasticity, the initial-boundary value problem of the governing differential equations can be reduced to a variational problem of the energy functional

$$\Pi = \int_{\Omega}[F - (f_i - c\dot{u}_i - \rho\ddot{u}_i)u_i]\mathrm{d}\Omega - \int_{\Gamma_t} T_i u_i \mathrm{d}\Gamma \tag{1}$$

i.e.,

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$$\delta\Pi = 0 \tag{2}$$

in which Ω is the domain of the body, F the strain energy density, u_i, $\dot{u}_i$ and $\ddot{u}_i$ the displacement, velocity and acceleration components respectively, f_i the body force, T_i the traction, ρ the mass density, c the damping of the material. Γ_t the boundary part on which the traction is given.

If u_i satisfies the deformation geometry condition

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{3}$$

and the displacement boundary condition

$$u_i = \bar{u}_i, \qquad (x, y, z) \in \Gamma_u \tag{4}$$

then the displacement field u_i satisfying the variational Eq.(2) must be the solution of the equations of motion

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = \rho\ddot{u}_i \tag{5}$$

under the initial value conditions

$$\begin{cases} u_i(x, y, z, 0) = g_i(x, y, z) \\ \dot{u}_i(x, y, z, 0) = h_i(x, y, z) \end{cases}, \qquad (x, y, z) \in \Omega \tag{6}$$

and the stress boundary conditions

$$\sigma_{ij}n_j = T_i, \qquad (x, y, z) \in \Gamma_t \tag{7}$$

in which ε_{ij} is the strain tensor, σ_{ij} the stress tensor, between which there is the relation

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{8}$$

Γ_u in Eq.(4) is the boundary part on which the displacement is given, $\bar{u}_i$, T_i, g_i and h_i in the above Eqs.(4), (6) and (7) are known functions, n_j the unit normal vector of the boundary, C_{ijkl} the elastic constants of the materials.

By discretizing the domain Ω into finite elements and introducing appropriate interpolation functions, Eq.(2) is reduced to the finite element equation as follows:

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{R(t)\} \tag{9}$$

in which $[M]$ is the total mass matrix, $[C]$ the damping matrix, $[K]$ the stiffness matrix, $\{\ddot{u}\}$, $\{\dot{u}\}$, $\{u\}$ and $\{R(t)\}$ the acceleration, velocity, displacement and equivalent force vectors of the nodes, respectively.

In what follows, Eq.(9) is solved by taking the Newmark integration step by step, i.e., instead integrating Eq.(9) every time only at the discrete points $t, t+\Delta t$, $t+2\Delta t$, $\cdots$, and within each interval Δt , the displacement, velocity and acceleration are assumed to satisfy certain relations. In this way and omitting the damping, Eq.(9) is reduced to at time $t+\Delta t$

$$[M]\{\ddot{u}_{t+\Delta t}\} + [K]\{u_{t+\Delta t}\} = \{R_{t+\Delta t}\} \tag{10}$$

and it is assumed that

$$\{\dot{u}_{t+\Delta t}\} = \{\dot{u}_t\} + \Delta t(1-\alpha)\{\ddot{u}_t\} + \Delta t\alpha\{\ddot{u}_{t+\Delta t}\}, \qquad 0 \leqslant \alpha \leqslant 1 \tag{11}$$

$$\{u_{t+\Delta t}\} = \{u_t\} + \Delta t\{\dot{u}_t\} + \frac{(\Delta t)^2}{2}(1-2\beta)\{\ddot{u}_t\} + (\Delta t)^2\beta\{\ddot{u}_{t+\Delta t}\}, \quad 0 \leqslant 2\beta \leqslant 1 \tag{12}$$

in which α and β are two parameters, the arithmetic is automatically stable as $\alpha \geqslant 0.5$, $\beta \geqslant 0.25(0.5+\alpha)^2$.

Based on the initial value condition, $\{u_0\}$ is known, and substituting this into Eq.(10) yields

$$[M]\{\ddot{u}_0\} = \{R_0\} - [K]\{u_0\} \tag{13}$$

from which the initial acceleration $\{\ddot{u}_0\}$ can be obtained. With the given $\{u_0\}$ and formulae (10), (11) and (12), the values of $\{u_{t+\Delta t}\}$, $\{\dot{u}_{t+\Delta t}\}$ and $\{\ddot{u}_{t+\Delta t}\}$ for the next interval Δt can be evaluated. Thus one can find the displacement, velocity and acceleration of every discrete point of time as well as the strains, stresses and stress intensity factor.

Ⅲ. THREE-DIMENSIONAL ELLIPTIC CRACK

As shown in Fig. 1, an elliptic crack with major and minor semi-axes a and b is in an infinite elastic medium, with the coordinate system set up in its center. The body is subjected to a dynamic tension $\sigma_{zz}(x, y, \pm\infty, t) = p_0 f(t)$ at infinity, in which p_0 = constant, $f(t)$ is an arbitrary function of time.

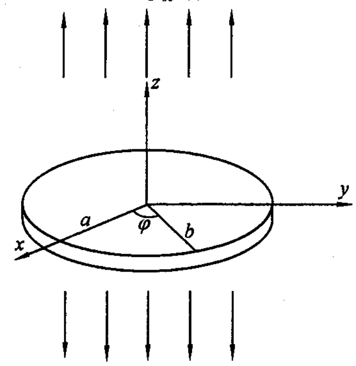

Fig. 1 The elliptic crack under impact.

For simplicity, it is assumed that the body is stress free at infinity, and the crack surface is subjected to a pressure, i.e. $\sigma_{zz}(x, y, 0, t) = -p_0 f(t)$ at $x^2/a^2 + y^2/b^2 \leqslant 1$. Because of the symmetry of the problem, it suffices to consider only a half space $z > 0$ (or $z < 0$), in which case there are the following initial and boundary conditions:

$$u_i(x, y, z, 0) = 0, \quad \dot{u}_i(x, y, z, 0) = 0 \tag{14}$$

$$\begin{aligned} \sigma_{ij} &= 0, & &\sqrt{x^2 + y^2 + z^2} \to \infty \\ \sigma_{zz} &= -p_0 f(t), & &z = 0,\ x^2/a^2 + y^2/b^2 \leqslant 1 \\ u_z &= 0, & &z = 0,\ x^2/a^2 + y^2/b^2 \geqslant 1 \\ \sigma_{xz} &= \sigma_{yz} = 0, & &z = 0,\ -\infty < x, y < \infty \end{aligned} \tag{15}$$

In the computation, let

$$p_0 f(t) = p_0 H(t) \tag{16}$$

where $H(t)$ is the Heaviside function.

Boundary condition (15) provides a higher symmetry, and it suffices to consider 1/8 space only, e.g. $x > 0$, $y > 0$, $z > 0$. In the computation, the dynamic three-dimensional singular element is used in the neighbourhood at the crack front[4]. The calculation by solving Eq. (10) determines the

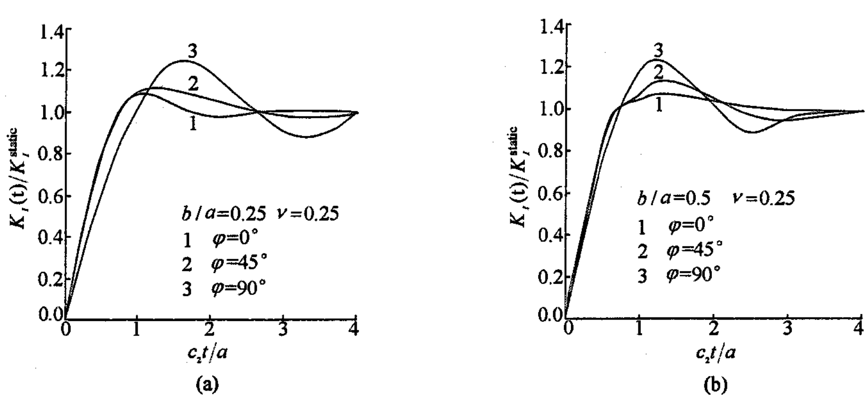

Fig. 2 Dynamic stress intensity factor versus time.

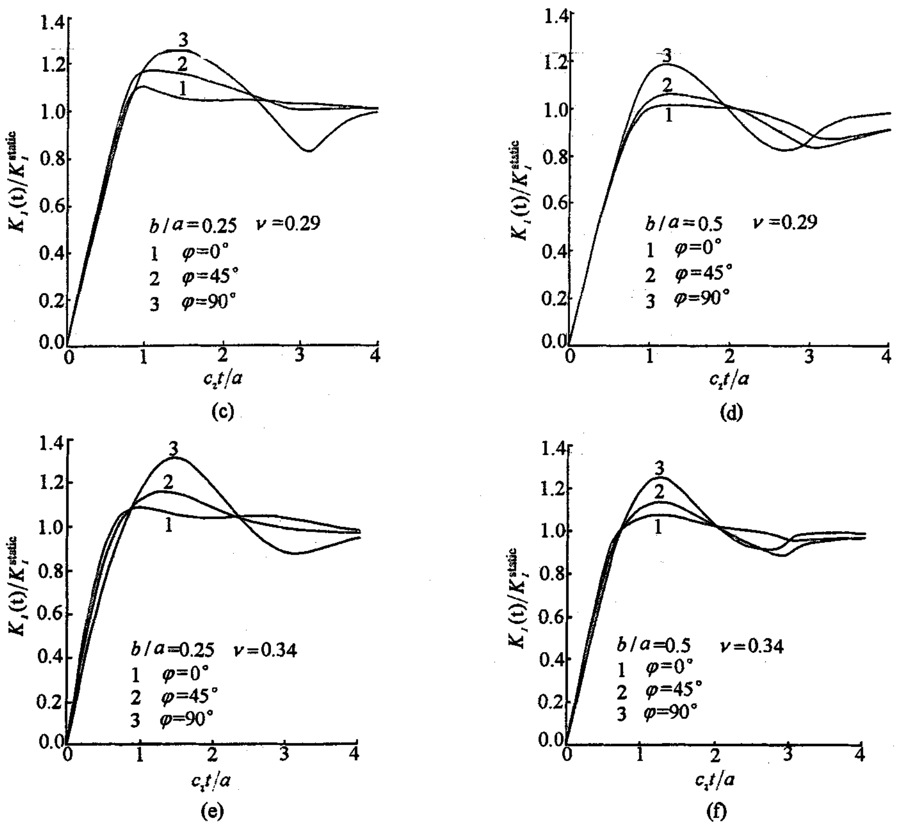

Fig.2 Dynamic stress intensity factor versus time.

displacement u_z for $z>0$, $x^2/a^2+y^2/b^2\leqslant 1$, i.e., the crack opening displacement $u_z(x,y,0,t)$. Let ε represent the distance measured from the front of the crack tip, φ the polar angle (i.e. $\varphi=\mathrm{tg}^{-1}(y/x)$ at plane $z=0$) (Fig.1). The crack opening displacement can be denoted by $u_z(\varepsilon,\varphi,0,t)$. In the vicinity of the crack tip, the stress state is in plane strain state, so it is well known that

$$u_z(\varepsilon,\varphi,0,t)=\frac{2(1-\nu^2)}{E\sqrt{\pi}}(2\varepsilon)^{1/2}K_I(b/a,\varphi,t) \tag{17}$$

in which E represents the Young's modulus, ν the Poisson's ratio of the material, and K_I is the stress intensity factor of model Ⅰ. From the values of $u_z(\varepsilon,\varphi,0,t)$ and Eq.(17), the dynamic stress intensity factor K_I is determined by taking the limit as $\varepsilon\to 0$.

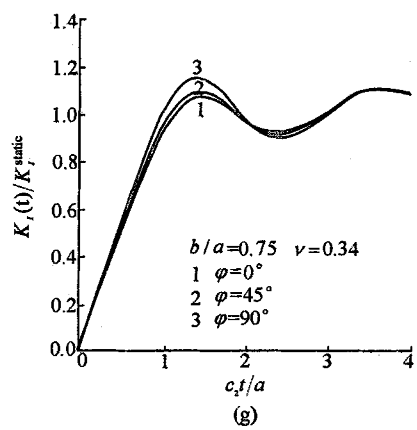

Fig.2 Dynamic stress intensity factor versus time.

The time dependence and variation of K_I versus the geometrical parameters are plotted in Fig.2, where normalized stress intensity $K_I(b/a,\varphi,t)/K_I^{static}$ and normalized time c_2t/a are used, in which

K_I^{static} is the well-known Green-Sneddon solution[2], i.e.

$$K_I^{\text{static}} = \frac{p_0\sqrt{\pi}}{E(k)}\left(\frac{b}{a}\right)^{1/2}(a^2\sin^2\varphi + b^2\cos^2\varphi)^{1/4} \tag{18}$$

and $E(k)$ is the complete elliptic integral of second kind, i.e. $E(k) = \int_0^{\pi/2}(1 - k^2\sin^2\theta)^{1/2}\mathrm{d}\theta$ with modulus of the ellipse $k = \left(\frac{a^2 - b^2}{a^2}\right)^{1/2}$.

Ⅳ. CONCLUSIONS AND DISCUSSION

The results given by Fig.2 show that the normalized dynamic stress intensity factors depend on the Poisson's ratio ν. The time variation of the stress intensity factors presents a similar diagram for different materials (whose Poisson's ratio values are from $\nu = 0.25$ to $\nu = 0.34$). The results may be used as universally effective values for conventional structural materials.

So far, it has been found that the numerical results of dynamic stress intensity factors of two-dimensional crack and penny-shaped cracks are special cases of $b/a \to 0$ and $b/a \to 1$ of the present work. The time-dependence of the dynamic stress intensity factors of different kinds of cracks shows similar behaviour. It can be seen that the stress intensity factors for different φ as $b/a \geqslant 0.75$ (Fig.2 (g)) are almost the same, indicating that the behaviour of the elliptic crack in the case is very close to that of the penny-shaped crack.

The approach adopted in this study is suitable not only for the dynamic crack problem but also for the dynamic contact problem. The results of the latter will be reported elsewhere.

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