Lifting of DG modules over DG algebras and a conjecture of Vasconcelos

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Introduction

(R, m, k) is assumed to be a local commutative noetherian ring with unity. When we say R is complete, we mean it is complete in m-adic topology.

Definition 1. Let R → S be a homomorphism of rings and let M be a finitely generated S-module. Then the finitely generated R-module L is called a lifting of M to R if

1. M ∼= S ⊗ R L;
2. Tori(S, L) = 0 for all integers i > 0.

The S-module M is said to be liftable to R, when such an R-module L exists.

Theorem 2. (Auslander, Ding and Solberg, 1993) Suppose R is complete and x = x1, ..., xn is an n-regular sequence in m. Let S = R/kR and M be a finitely generated S-module. If Exti(S, M) = 0, then M is liftable to R.

Definition 3. The finitely generated R-module C is semidualizing if it satisfies the following:
1. The homothety map C R HomR(C, C) given by ϕ ↦ rϕcomm is an isomorphism, and
2. Exti(R, C) = 0 for all i > 0.

Conjecture 4. (W. V. Vasconcelos, 1974) The number of isomorphism classes of semidualizing modules over a Cohen-Macaulay local ring is finite.

Theorem 5. (L. Christensen and S. Sather-Wagstaff, 2008) If K is a Cohen-Macaulay and equiprimary, then the number of isomorphism classes of semidualizing modules is finite.

Sketch of Proof. Pass to the completion of R to assume that R is complete. Let x be a system of parameters for R; this is a maximal regular sequence. Using the Theorem of Auslander, Ding and Solberg, there is a one-to-one correspondence between the set of isomorphism classes of semidualizing R-modules and that of kR-modules. So we pass to the quotient kR to assume that R is artinian. Since R is equiprimary and artinian, Cohen’s structure theorem implies R is a k algebra. Now the theorem follows from a result in representation theory.

General form of Vasconcelos’ Conjecture. The number of isomorphism classes of semidualizing modules over a local ring is finite.

To answer this question, we consider Differential graded algebras and Differential graded modules (DG algebras and DG modules).

DG Algebras And DG Modules

Complexes of R-modules (R-complexes for short) are indexed homologically:

and in M, we write |m| = i.

Definition 6. A commutative differential graded algebra over R (DG R-algebra for short) is a bounded below (Δn ∈ 0 for n < 0) R-complex A equipped with a chain map μ: A ⊗ R A → A denoted μn(A ⊗ n, b) = ab (which is the product of the complexes) that is
1. Associative: for all a, b, c ∈ A we have ab(c) = a(bc);
2. Unital: there is an element 1 ∈ A such that for all a ∈ A, we have 1·a = a;
3. Graded Commutative: for all a, b ∈ A we have a·b = (−1)|a||b|ba and a·a = 0 when |a| is odd.

The fact that the product on Δ is a chain map says that μ0 satisfies the Leibniz rule:

where [a, b] = ab − ba.

Example 7. For elements α1, ..., αn ∈ A, the Koszul complex K = K(α1, ..., αn) is the exterior algebra on αi, considered as an R-complex

K = 0 → K1 → K2 → ··· → Kn → 0

be a finite dimensional commutative DG algebra over F. Let W = K(α1, ..., αn) be a graded module with (deg(Wi))i=1 be r, for every 0 ≤ i ≤ n. Let r = (r0, ..., rn) and M = ⊗i ri Md be the set of all DG A-module structures on W. Md is the set of rational points of an algebraic scheme ModA,F over F, which is described in the functorial point of view as follows; for any finite dimensional commutative DG algebra A over F, we have

where (Δi ⊗ Δj) − (Δj ⊗ Δi) is the commutator of Δi and Δj.

Theorem 14. There are only finitely many isomorphism classes of semidualizing DG A-modules with the fixed dimension vector r = (r0, ..., rn).

Sketch of Proof. It can be seen that the orbit of every DG A-module in ModA,F under ModA,F is an open subscheme of ModA,F. Let (1, 0)n ⊆ A be the set of these open sets, which are disjoint. (1, 0)n is an open cover for the open subset of ModA,F. ModA,F has a finite discrete covererve. Since the open sets (1, 0)n are disjoint, A is a finite set.

Theorem 15. Let B be a DG R-algebra such that each Bk is free over R of finite rank. For an element t ∈ m, let (1, 0)n ⊆ B be the Koszul complex 0 → K1 → K2 → ··· → Kn and let D denote the DG R-algebra K(1, 0)n ⊗ R B. If R is complete and N is a semifree DG D-module such that ExtR(N, N) = 0, then N is liftable to B.

Theorem 16. Let B be complete and assume that B and D are the DG R-algebras considered in Theorem 15. Let N be a semifree DG D-module such that ExtR(N, N) = 0 for some even integer d. If M is a lifting of N to D, then ExtD(M, M) = 0.

Theorem 17. Suppose that R is complete and t = t1, ..., tn is a sequence of elements of m. If N is a semidualizing DG module over R, then there exists a semidualizing R-complex M which is a lifting of N to R.

Sketch of Proof. We can replace N by a semifree DG resolution over R and assume that N is a semifree DG module. The existence of a complex M which is a lifting of N to R follows by induction on the previous two theorems. But note that

where the last isomorphism is a special case of tensor-equivalence. Therefore, R ∼= ExtR(N, N).

Notation 18. Let A be a commutative local DG algebra, Ω(A) (resp. Ω(R)) denotes the set of shift-iso or morphism classes of semidualizing DG A-modules (resp. semidualizing R-complexes).

Corollary 19. There is a one-to-one onto map from the set Ω(R) to the set Ω(R).

Theorem 20. There are only finitely many semidualizing R-complexes up to shift isomorphism.

Sketch of Proof. Without loss of generality we may assume R is complete and k is algebraically closed.

Let a = a1, ..., an be a minimal set of generators for R and set K = K(a1). We know that K ∼= T where T is of the form

where each ai is a noninvertible element. In fact T is a finite dimensional DG algebra over k. Now by Corollary 19 and Fact 1(12) we get the following equivalences

Now suppose that C is a semidualizing R-complex and denote by Ĉ the semidualizing DG T-module corresponding to C. Without loss of generality we can assume that int(C) = 0. By using a minimal semifree DG resolution, without loss of generality we can suppose that Ĉ is a semifree DG T-module. We can also replace Ĉ by its left truncation Ω(C) and R and suppose that

where for every i, Ĉi = Ω(Ci) and Ω ≥ i. Now it can be seen that

where the right-hand side is a constant. It is obvious that there are only finitely many (r0, ..., rn) ∈ N+ such that

with

Therefore, Theorem 14 implies that |Ω(R)| is finite and hence |Ω(R)| is finite.