K.N.Rao Contest 2012

0. (A warm-up) Two volumes of Sherlock Holmes Adventures are standing on a book shelf. The thickness of the pages in each book is 1.5 inches. The thickness of each cover 0.1 of an inch. A worm digs through the book from page 1 of Volume 1 to the last page of Volume 2. What distance does the worm dig?

   Solution: 0.2 of an inch.

1. Which is larger $\log_8 9$ or $\log_9 10$?

   Solution: Using the properties of the log function:

$$\log_n(n+1) = \log_n n\left(1+\frac{1}{n}\right) = 1+\log_n\left(1+\frac{1}{n}\right) \geq$$
$$\geq 1+\log_n\left(1+\frac{1}{n+1}\right) \geq 1+\log_{n+1}\left(1+\frac{1}{n+1}\right) = \log_{n+1}(n+2)$$

2. Let $c_0, c_1, \ldots, c_n$ be constants such that

$$c_0+\frac{c_1}{2}+\frac{c_2}{3}+\ldots,\frac{c_n}{n+1}=0.$$

   Show that the equation

$$c_0+c_1x+\ldots+c_nx^n=0$$

   has a real solution on the interval $(0,1)$.

   Solution I: Integrating the equation from 0 to 1 and using the condition, one has that

$$\int_0^1 f(x)\mathrm{d}x = 0,$$

   which for continuous f implies that $f(x)$ has a root $0 \leq x \leq 1$.

   Solution II: Consider function

$$g(x)=c_0x+\frac{c_1x^2}{2}+\frac{c_2x^3}{3}+\ldots,\frac{c_nx^n}{n+1}.$$

   We have $g(0)=0$, $g(1)=0$, implying that there is a point $x \in [0,1]$ at which $g'(x)=0$.

3. Calculate

$$\sum_{n=1}^{9999}\frac{1}{(\sqrt{n+1}+\sqrt{n})(\sqrt[4]{n}+\sqrt[4]{n+1})}$$

   Solution: Using the fact that

$$\frac{\sqrt[4]{n+1}-\sqrt[4]{n}}{(\sqrt{n+1}+\sqrt{n})(\sqrt[4]{n}+\sqrt[4]{n+1})(\sqrt[4]{n+1}-\sqrt[4]{n})}=\sqrt[4]{n+1}-\sqrt[4]{n}$$

   we obtain a telescopic series, all the terms of which cancel except for the first and last. The final answer is

$$-1+\sqrt[4]{10000}=9.$$

4. Do there exist at least three 10 digit numbers, divisible by 11, which are written using each digit $0, 1, \ldots, 9$ only once?

   Solution: Yes. An integer is divided by 11 if and only if the difference between the sum of the digits at the even positions and the sum of the digits at the odd positions is divided by 11. Consider the number 9876543210. The difference is 5. If we exchange 8 and 5 we obtain one of the desired numbers. Similarly, it is possible to exchange 7 and 4, and 6 and 3. Q.E.D.

5. Solve the linear system

$$\begin{aligned} x_1 + x_2 + \ldots + x_{n-2} + x_{n-1} &= n \\ x_1 + x_2 + \ldots + x_{n-2} + x_n &= n-1 \\ &\ldots \\ x_1 + x_3 + \ldots + x_{n-1} + x_n &= 2 \\ x_2 + x_3 + \ldots + x_{n-1} + x_n &= 1 \end{aligned}$$

   Solution: Summing all the equations we find

$$\sum_{i=1}^{n} x_i = \frac{n(n+1)}{2(n-1)} .$$

   Subtracting from this all the equations we have

$$x_k = \frac{n(n+1)}{2(n-1)} - k, \quad k = 1, \ldots, n.$$

6. Let S be any collection of 9 distinct points in the space such that each point has integer coordinates. Prove that there is a line segment with endpoints both in S such that the midpoint of the segment also has integer coordinates.

   Solution: If we are given $2n-1$ integers, at least n are odd or even simultaneously. Out of 9 we always have 5 points whose abscissas have the same evenness; out of them there are 3 points with ordinates with the same evenness, out of which there are 2 points with applicates with the same evenness. Q.E.D.

7. Find

$$\lim_{n\to\infty} \frac{1 + 2^2 + 3^3 + \ldots + n^n}{n^n}$$

   Solution:

$$1 \le \frac{1 + 2^2 + 3^3 + \ldots + n^n}{n^n} \le \frac{n + n^2 + n^3 + \ldots + n^n}{n^n} = \frac{n(n^n - 1)}{(n-1)n^n} .$$

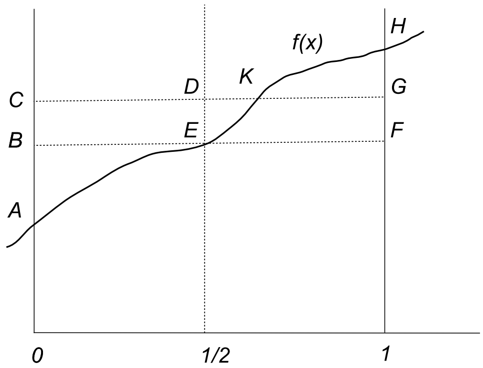
   Since the right hand side tends to 1, the answer is 1.

8. For which C is the integral

$$\int_0^1 |f(x) - C| \, \mathrm{d}x$$

   minimal if $f(x)$ is continuous and increasing on $[0, 1]$?

   Solution: Assume that $C > f(1/2)$. Then, from the figure,



it follows that

$$\int_0^1 |f(x) - C|\,\mathrm{d}x = S_{ACK} + S_{KHG} = S_{ABE} + S_{BCKE} + S_{EKHF} - S_{EKGF} >$$

$$> S_{ABE} + S_{EKHF} = \int_0^1 |f(x) - f(1/2)|\,\mathrm{d}x.$$

Similar argument shows that $C < f(1/2)$ does not work. Hence $C = f(1/2)$.

9. Find all 2×2 matrices $\boldsymbol{A}$ with non-zero elements such that $\boldsymbol{A}^2 = 0$.

Solution: If

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then $\boldsymbol{A}^2 = 0$ is equivalent to a system of four equations, which can be solved. Finally

$$\boldsymbol{A} = \begin{bmatrix} a & -a^2/c \\ c & -a \end{bmatrix}, \quad a, c \in \mathbb{R}\backslash\{0\}.$$