K.N.Rao Contest 2013

0. (*3pts*) (A warm-up) Imagine a chess board of 64 squares with two corner squares removed, hence total of 62 squares. Can you cover this board with dominos, each of which covers exactly two adjacent squares, such that there are no not covered squares and no square is covered by more than one domino?

   Solution: The answer depends on which corners are removed. If the corners are on the same side, then the answer is "yes." If these two corners are on the opposite sides, then "no" because in this case we remove the squares of the same color, but each domino has to cover squares of different colors.

1. (*5pts*) Which is larger
$$\sqrt{2}^{\sqrt{3}} \text{ or } \sqrt{3}^{\sqrt{2}}?$$
   Solution: Raise both sides to the power $2\sqrt{2}$:
$$2^{\sqrt{6}} < 2^3 = 8 < 9,$$
   which proves that the first one is smaller.

2. (*8 pts*) Consider the operations of union, intersection, and difference on sets X and Y. By definition, the union $X \cup Y$ of two sets X and Y is the set that consists of the elements that belong to X, or belong to Y, or to both X and Y. The intersection $X \cap Y$ of the two sets X and Y is the set that consists of the elements that belong to X and Y. The difference $X \setminus Y$ is the set that consists of the elements that belong to X and do not belong to Y. It is possible to express some operations through the others. For example, convince yourself that $X \cap Y = X \setminus (X \setminus Y)$.

   Is is possible to express the difference through the union and the intersection?

   Solution: The answer is no. Consider the sets $X = \{1, 2\}$ and $Y = \{2\}$. Operations of intersection and union always result either in X or in Y:
$$X \cup X = X \cap X = X \cup Y = Y \cup X = X,$$
$$Y \cup Y = Y \cap Y = X \cap Y = Y \cap X = Y,$$
   whereas we need to get $X \setminus Y = \{1\}$.

3. (*6pts*) A graph G is a non-empty set of vertices $V(G)$ together with the set of edges $E(G)$ of 2 element unordered subsets of $V(G)$. Think about geometric pictures with vertices as points and edges as lines connecting some of the points. The *degree* of a vertex v is the number of the edges adjacent to v. The total number of vertices if called *the order* of the graph. A graph G of order $p \geq 2$ is *perfect* if no two of its vertices have equal degrees. Prove that *no graph is perfect.*

   Solution: First, no vertex can have degree more then the order of G minus 1. Since we are required that all the vertices have different degrees, we are given no choice as to have the degrees $0, 1, \ldots, p-1$. Assume that we have such a graph. Now pick a vertex that has degree $p - 1$, which means that this vertex is connected to all other vertices, hence no vertex can have degree zero, contradiction.

4. (*10pts*) Prove that
$$\left(\frac{\sin x}{x}\right)^3 > \cos x, \quad 0 < x < \frac{\pi}{2}.$$
   Solution: Consider an equivalent inequality
$$\frac{\sin x}{\sqrt[3]{\cos x}} > x.$$

Since at the point $x = 0$ the left hand side equals to the right hand side, it is enough to prove that the derivative of the left hand side is bigger than the derivative of the right hand side (after some rearrangement):

$$\frac{2\cos^2 x + 1 - 3\cos^{4/3} x}{3\cos^{4/3} x} > 0,$$

or, equivalently

$$2u + 1 - 3u^{2/3} > 0, \quad 0 < u < 1, \quad u = \cos^2 x.$$

The last inequality can be proved by noting that the function $3u^{2/3}$ is strictly increasing on $(0, 1)$ and concave, and having the derivative 2 at the point $u = 1$ (make a graph).

5. (*7pts*) Consider the polynomial

$$p(x) = x^3 + 2x^2 + 3x + 4,$$

and let

$$s_n = x_1^n + x_2^n + x_3^n,$$

where $x_i, i = 1, 2, 3$ are the roots of $p(x)$. Show that

$$s_1 = s_2 = s_3.$$

Solution: Vieta's theorem implies

$$x_1 + x_2 + x_3 = -2, \quad x_1x_2 + x_2x_3 + x_1x_3 = 3, \quad x_1x_2x_3 = -4.$$

Now

$$s_2 = (x_1 + x_2 + x_3)^2 - 2(x_1x_2 + x_2x_3 + x_1x_3) = -2.$$

We also have

$$x_i^3 = -2x_i^2 - 3x_i - 4 \implies s_3 = -2s_2 - 3s_1 - 12 = -2.$$

6. (*8pts*) Find the limit

$$\lim_{n\to\infty} \cos\frac{\pi}{4}\cos\frac{\pi}{8}\ldots\cos\frac{\pi}{2^{n+1}}.$$

Solution: We have

$$\cos\frac{\pi}{4}\cos\frac{\pi}{8}\ldots\cos\frac{\pi}{2^{n+1}} = \frac{\cos\frac{\pi}{4}\cos\frac{\pi}{8}\ldots\cos\frac{\pi}{2^{n+1}}\sin\frac{\pi}{2^{n+1}}}{\sin\frac{\pi}{2^{n+1}}} = \frac{\sin\frac{\pi}{2}}{2^n\sin\frac{\pi}{2^{n+1}}},$$

which implies that the limit is

$$\frac{2}{\pi}.$$

7. (*6pts*) The numbers 53295, 67507,88825, 81719, 39083 are divided by 3553. Without calculating the determinant of the matrix

$$A = \begin{bmatrix} 5 & 3 & 2 & 9 & 5 \\ 6 & 7 & 5 & 0 & 7 \\ 8 & 8 & 8 & 2 & 5 \\ 8 & 1 & 7 & 1 & 9 \\ 3 & 9 & 0 & 8 & 3 \end{bmatrix}$$

show that it is divided by 3553.

Solution: Multiply the first column by 10000, second by 1000, third by 100, fourth by 10 and add to the last one. The result will be

$$\begin{bmatrix} 5 & 3 & 2 & 9 & 53295 \\ 6 & 7 & 5 & 0 & 67507 \\ 8 & 8 & 8 & 2 & 88825 \\ 8 & 1 & 7 & 1 & 81719 \\ 3 & 9 & 0 & 8 & 39083 \end{bmatrix}$$

and the determinant did not change. Now we can factor 3553 from the last column, which means that $\det A$ is divided by 3553.

8. (*8pts*) Find all the differentiable on $\mathbb{R}$ functions, that for any $x, y \in \mathbb{R}$ satisfy

$$f(x+y) = f(x) + f(y).$$

Solution: First put $y = 0$, which implies that $f(0) = 0$. Take the derivative with respect to x considering y a constant:

$$f'(x+y) = f'(x),$$

from where $f'(x) = \text{const} = k$. Finally,

$$f(x) = kx.$$