K.N.Rao Contest 2014

0. (*3pts*) (A warm-up) A water lily divides every day into two exact copies. Hence there is 1 lily on the first day, 2 lilies on the second day, 4 lilies on the third day, and so on. If there is initially only one lily in the pond then the surface of the pond will be covered by the lilies on the 30th day. How long will it take to cover the surface of the pond if there are two lilies initially?

   Answer: On the 29th day the surface of the pond will be covered by lilies.

1. (*3pts*) (Another warm-up) Prove

$$\ln\tan 1^\circ \cdot \ln\tan 2^\circ \cdot \ldots \cdot \ln\tan 88^\circ \cdot \ln\tan 89^\circ = 0.$$

2. (*5pts*) Prove that if A and B are two non zero square matrices, and

$$AB = O,$$

   where O is the zero matrix (the matrix whose all elements are equal to zero), then $\det A = \det B = 0$.

   Solution: Assume that $\det A \neq 0$, then there exists A^{-1}. Multiply the equality $AB = O$ by A^{-1} from the left:

$$A^{-1}AB = A^{-1}O \Longrightarrow B = O,$$

   which contradicts the initial condition. The same for B.

3. (*6pts*) Find the limit

$$\lim_{x\to\infty} \frac{(ax+1)^n}{x^n+b}, \quad n \in \mathbf{Z}, \quad a, b \in \mathbf{R}.$$

   Solution: Consider first $n \in \mathbf{N}$. Then

$$\lim_{x\to\infty} \frac{(ax+1)^n}{x^n+b} = \lim_{x\to\infty} \frac{x^n\left(a+\frac{1}{x^n}\right)^n}{x^n\left(1+\frac{b}{x^n}\right)} = a^n.$$

   If $n = 0$ then the limit is $(1+b)^{-1}$.

   Now let $n = -k$, $k \in \mathbf{N}$, then

$$\lim_{x\to\infty} \frac{(ax+1)^n}{x^n+b} = \lim_{x\to\infty} \frac{x^k}{(1+bx^k)(ax+1)^k} = \begin{cases} 0, & ab \neq 0, \\ a^{-k} = a^n, & a \neq 0, b = 0, \\ b^{-1}, & a = 0, b \neq 0, \\ \infty, & a = b = 0. \end{cases}$$

4. (*6pts*) On the plane we have 3 points $A(3-t, 6+2t)$, $B(1+t, 3-t)$, $C(t-1, 1)$. For which values of t point C is not visible from point A?

   Solution: All three points have to be on the same line, and B must be between A and C. This will be true if $\overrightarrow{AB} = \lambda\overrightarrow{BC}$ for some $\lambda > 0$, which implies

$$\frac{2t-2}{-2} = \frac{-3-3t}{t-2} > 0,$$

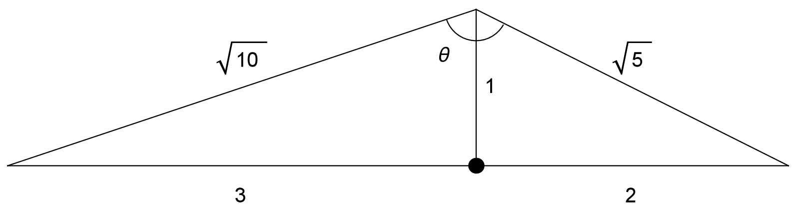
   which has the only solution $t = 3 - \sqrt{10}$.

5. (*7pts*) 2013 was the year of π. Indeed,

$$\arctan 2 + \arctan 0 + \arctan 1 + \arctan 3 = \pi.$$

Prove this equality. (Here $\arctan x$ is the function inverse to $\tan x$, $x \in (-\pi/2, \pi/2)$, another common notation is $\tan^{-1} x$.)

Solution: We have $\arctan 0 = 0$ and $\arctan 1 = \frac{\pi}{4}$. Consider the following triangle:



By construction, we have that $\theta = \arctan 2 + \arctan 3$. By cosine theorem,

$$(2+3)^2 = (\sqrt{5})^2 + (\sqrt{10})^2 - 2\sqrt{5}\sqrt{10}\cos\theta,$$

or

$$\cos\theta = -\frac{1}{\sqrt{2}} \implies \theta = \frac{3\pi}{4},$$

which concludes the proof.

Remark: Another way to prove the equality is to start with

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}.$$

6. (*5pts*) Find an equation of the straight line passing through $M(3,4)$ and forming together with the coordinate axes a triangle whose area is 3 square units.

Solution: It is convenient to look for the line equation in the form

$$\frac{x}{a} + \frac{y}{b} = 1.$$

By the initial conditions and the geometric meaning of a and b (the intervals on the axes cut by the line) we find a system of two equations

$$\frac{3}{a} + \frac{4}{b} = 1, \quad |ab| = 6,$$

which has solutions

$$(3/2, -4), \quad (-3, 2).$$

Therefore, two equations that solve our problem are

$$y = \frac{8}{3}x - 4, \qquad y = \frac{2}{3}x + 2.$$

7. (*5pts*) Evaluate the integral

$$\int_0^\pi \sqrt{1 + \cos 2x}\, \mathrm{d}x.$$

Solution: Since $1 + \cos 2x = 2\cos^2 x$ then

$$\int_0^\pi \sqrt{1+\cos 2x}\, \mathrm{d}x = \sqrt{2}\int_0^\pi |\cos x|\, \mathrm{d}x = 2\sqrt{2}.$$

8. (*8pts*) Find all integer solutions to

$$\sqrt{x+\sqrt{x}} = y - 2002.$$

Solution: The equation is equivalent to

$$x + \sqrt{x} = (y - 2002)^2, \quad y \geq 2002.$$

Since x is integer then, from the first equality, $\sqrt{x}$ is also integer. For the quadratic equation

$$t^2 + t - (y - 2002)^2 = 0$$

to have integer roots it is necessary that the discriminant

$$D = 1 + 4(y - 2002)^2$$

be a full square. We have that $D = d^2 = 1 + b^2$, where b is integer, therefore $b = 0$, or $y = 2002$, which implies that $t = 0$ and $t = -1$, or $x = 0$. The final answer is $(0, 2002)$ and no other solutions exist.

9. (*10pts*) For each pair of real numbers a and b consider the sequence $(p_n)_{n=0}^{\infty}$ of real numbers defined as

$$p_n = \lfloor 2\{an + b\} \rfloor.$$

Here $\{a\}$ is the fractional part of the number a, $0 \leq \{a\} < 1$, and $\lfloor a \rfloor$ is the integral part of the number a, such that $a = \lfloor a \rfloor + \{a\}$. Any k successive elements of (p_n) are called a *word*. Is it true that *any* ordered collection of zeroes and ones of length k is a word of some sequence (p_n) defined by some a and b for (i) $k = 4$, (ii) $k = 5$?

Solution: First, any sequence (p_n) defined as above consists of zeros and ones because $2\{a\}$ is a number from 0 (included) to 2 (not included). Consider a geometric interpretation of this problem. Let $\{a\}$ be the point on the unit circle rotated $2\pi\{a\}$ counter-clockwise from the initial point. Then we get a sequence of points on the circle

$$\{b\},\ \{b + a\},\ \{b + 2a\},\ \ldots$$

If the point $\{b + na\}$ in on the upper half of the circle, then $p_n = 0$, otherwise $p_n = 1$. Next, note that $\{a - b\} = \{\{a\} - \{b\}\}$ therefore, each next point on the circle is obtained from the previous by $2\pi\{a\}$ rotation. Therefore, if one has a lot of points in the upper half of the circle (this means that we have a lot of zeroes in our sequence), then it means that the angle of rotation $2\pi\{a\}$ is small, and we cannot have a lonely one and then zeros again, we will have to spend some time in the lower half of the circle. In particular, for $k = 5$ the word 00010 is impossible since it means that $\{a\}$ is less than 1/4 of the full rotation due to the presence of 000. On the other hand, the part 010 means that the sequence was in the upper half, then in the lower half, and then in the upper half again, which means that $\{a\}$ is bigger than 1/4 of the full rotation. Hence the answer for (ii) is no. For (i) all 16 words are possible, which can be checked by direct analysis of each case.