K.N.Rao Contest 2015

0. (*3pts*) (A warm-up) There are 9 perfunctory indistinguishable coins, one of which is fake. It is known that the fake coin is lighter than the real ones. Is it possible to find the fake coin using only two weighings on a weights with two plates? Please explain how if the answer if positive.

   Solution: Take 6 coins, three on one plate and three on another. If the weights are even, then the fake coin is among the remaining three. Pick two, if the coin is among them, you will see it on the weights. If the weights are even, then the fake coin is the one that left. If in the first attempt one of the plates is lighter than another then the fake coin is among these three.

1. (*6pts*) Prove that
$$\cos^2 x \sin x > -0.(6), \quad x \in [-\pi, \pi],$$
   where 0.(6) means 0.666666....

   Solution: We need to find the minimum of $f(x) = \cos^2 x \sin x = \sin x - \sin^3 x$ on $[-\pi, \pi]$. We find the derivative and critical points that are the roots of $\cos x(1 - 3\sin^2 x) = 0$. This equation has the roots
$$x = \pm\frac{\pi}{2}, \quad x = \pm \arcsin\frac{1}{\sqrt{3}}, \quad x = \pi - \arcsin\frac{1}{\sqrt{3}}, \quad x = \arcsin\frac{1}{\sqrt{3}} - \pi.$$
   Evaluating f at these points we find $\min_{x\in[-\pi,\pi]} f(x) = -\frac{2}{3\sqrt{3}} > -\frac{2}{3} = -0.(6)$.

2. (*5pts*) Let ABC be a triangle. Let $\vec{a} \cdot \vec{b}$ denote the usual scalar product of vectors $\vec{a}$ and $\vec{b}$. Prove that
$$\overrightarrow{AB} \cdot \overrightarrow{BC} + \overrightarrow{BC} \cdot \overrightarrow{CA} + \overrightarrow{CA} \cdot \overrightarrow{AB} < 0.$$
   Solution: Let $\vec{a} = \overrightarrow{AB}$, $\vec{b} = \overrightarrow{BC}, \vec{c} = \overrightarrow{CA}$. Using the fact that $\vec{a} + \vec{b} + \vec{c} = 0$ and hence $(\vec{a} + \vec{b} + \vec{c})^2 = 0$ we obtain the desired inequality.

3. (*5pts*) Find the number of real solutions to
$$xe^{-x} + e^{-x} + \frac{x^2}{2} - 1 = 0.$$
   Solution: Since $(xe^{-x} + e^{-x} + \frac{x^2}{2} - 1)' \geq 0$ hence this equation can have at most one solution, which can be just guessed: $x = 0$.

4. (*7pts*) Find an equation of the tangent line to the graph of $f(x) = (6x + 7)^{3/2} - 9x + 4$ if it is known that on this line there are no points with equal coordinates.

   Solution: Since the tangent line does not intersect the graph of $g(x) = x$ therefore we need to solve $f'(x) = 1$ which implies that $x = -\frac{467}{486}$ and hence the equation of the tangent line is
$$y = x + \frac{6269}{162}.$$

5. (*8pts*) Find the sum
$$1^3 + 3^3 + 5^3 + \ldots + (2n - 1)^3.$$
   Solution: Induction and several observations lead to
$$1^3 + 2^3 + \ldots + m^3 = \left(\frac{m(m + 1)}{2}\right)^2.$$

Put $m = 2n$ and find that

$$1^3 + 2^3 + \ldots + (2n-1)^2 + (2n)^3 = \left(\frac{2n(2n+1)}{2}\right)^2.$$

The left hand side can be transformed into

$$1^3 + 3^3 + \ldots + (2n-1)^2 + 2^3(1^3 + 2^3 + \ldots + n^3),$$

which leads to the final answer

$$\left(\frac{2n(2n+1)}{2}\right)^2 - 2^3\left(\frac{n(n+1)}{2}\right)^2 = n^2(2n^2-1).$$

6. (*9pts*) In how many parts the diagonals of a convex polygon with n vertexes divide this figure if we know that neither three of them intersect at the same point.

   Solution: Each time we draw a new diagonal the number of parts increases by $m+1$, where m is the number of intersection points of the new diagonal with the previous ones, in other words every new diagonal and every new intersection point increase the number of parts by 1. Therefore, the total number of parts is $D + P + 1$, where D is the number of diagonals, P is the number of intersection points. By counting argument $D = n(n-3)/2$. Each intersection point is determined by 4 vertices of the polygon, hence

$$P = \binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{24}.$$

7. (*6pts*) Prove that the polynomial

$$P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n,$$

   equal at $x = 0$ and $x = 1$ to odd values, cannot have integer roots.

   Solution: If x an even number then $P(x) \equiv a_n \pmod 2$, if x an odd number then $P(x) = a_0 + \ldots + a_n \pmod 2$, in both cases these numbers are odd and hence cannot be equal to zero.

8. (*7pts*) Find all integer solutions to

$$x + y = x^2 - xy + y^2.$$

   Solution: Note that

$$(x-y)^2 + (x-1)^2 + (y-1)^2 = 2(x^2 - xy + y^2 - x - y) = 2,$$

   and hence we can only have solutions $(0,0), (2,2), (1,0), (1,2), (0,1), (2,1)$.

9. (*9pts*) Solve the system

$$x + y = a,$$
$$x^5 + y^5 = b^5.$$

   Solution: Let $xy = t$. Then

$$x^5 + y^5 = (x+y)^5 - 5(x+y)^3xy + 5(x+y)x^2y^2 = a^5 - 5a^3t + 5at^2.$$

   If $a \neq 0$ then the solutions to it are given by

$$t = \frac{1}{2}\left(a^2 \pm \sqrt{\frac{a^5 + 4b^5}{5a}}\right).$$

   We reduced our problem to

$$x + y = a, \quad xy = t,$$

   whose solution can be reduced to the quadratic equation.