A local ring has only finitely many semidualizing modules up to isomorphism

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Notation

$\mathfrak{S}(R) = \{$isomorphism classes of semidualizing R-modules$\}$.

A Conjecture and Partial Solution

Fact (Base-change)

If $R \to S$ is a local homomorphism of finite flat dimension, then $\mathfrak{S}(R) \hookrightarrow \mathfrak{S}(S)$ by $C \mapsto S \otimes_R C$.

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A result of Happel essentially shows that $\mathfrak{S}(R'/(\mathbf{x}))$ is finite. □

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Example (The Koszul complex)

$K = K^R(\mathbf{x})$ is a DG R-algebra for each sequence $\mathbf{x} \in R$.

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Theorem (Nasseh and Sather-Wagstaff '12)

The sets $\mathfrak{S}(R)$ and $\mathfrak{S}_{dg}(R)$ are finite.

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$\widetilde{K}$ is a minimal Q-free resolution of $\overline{k}$.

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There is a flat local ring homomorphism $R \to (R', \mathfrak{m}R', \overline{k})$ such that R' is complete.
Let $\mathbf{x} \in \mathfrak{m}R'$ be minimal generating sequence and $K = K^{R'}(\mathbf{x})$.
Let Q be a regular local ring surjecting onto R'.
Let $\widetilde{\mathbf{x}} \in Q$ be a lift of $\mathbf{x}$, and set $\widetilde{K} = K^Q(\widetilde{\mathbf{x}})$.
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$\widetilde{K}$ is a minimal Q-free resolution of $\overline{k}$.

Theorem (Nasseh and Sather-Wagstaff '12)

The sets $\mathfrak{S}(R)$ and $\mathfrak{S}_{dg}(R)$ are finite.

Outline of Proof.

It suffices to prove that $\mathfrak{S}_{\mathrm{dg}}(R)$ is finite since $\mathfrak{S}(R) \hookrightarrow \mathfrak{S}_{\mathrm{dg}}(R)$.

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Solution to Vasconcelos' Conjecture

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$A \otimes_Q \overline{k}$ is a finite dimensional DG $\overline{k}$-algebra, and $\mathfrak{S}_{\mathrm{dg}}(A \otimes_Q \overline{k})$ is finite. □