Lifting of DG modules over DG algebras and a conjecture of Vasconcelos

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Introduction

$(R, \mathfrak{m}, k)$ is assumed to be a local commutative noetherian ring with unity. When we say R is complete, we mean it is complete in $\mathfrak{m}$-adic topology.

Definition 1. Let $R \to S$ be a homomorphism of rings and let M be a finitely generated S-module. Then the finitely generated R-module L is called a *lifting* of M to R if

1. $M \cong S \otimes_R L$, and
2. $\operatorname{Tor}_i^R(S, L) = 0$ for all integers $i > 0$.

The S-module M is said to be *liftable* to R, when such an R-module L exists.

Theorem 2. (Auslander, Ding and Solberg, 1993) Suppose R is complete and $\mathbf{x} = x_1, \cdots, x_n$ is an R-regular sequence in $\mathfrak{m}$. Let $S = R/\mathbf{x}R$ and M be a finitely generated S-module. If $\operatorname{Ext}_S^2(M, M) = 0$, then M is liftable to R.

Definition 3. The finitely generated R-module C is *semidualizing* if it satisfies the following:

1. The homothety map $\chi_C^R \colon R \to \operatorname{Hom}_R(C, C)$ given by $\chi_C^R(r)(c) = rc$ is an isomorphism, and
2. $\operatorname{Ext}_R^i(C, C) = 0$ for all $i > 0$.

Conjecture 4. (W. V. Vasconcelos, 1974) The number of isomorphism classes of semidualizing modules over a Cohen-Macaulay local ring is finite.

Theorem 5. (L. W. Christensen and S. Sather-Wagstaff, 2008) If R is *Cohen-Macaulay* and *equicharacteristic*, then the number of isomorphism classes of semidualizing modules is finite.

Sketch of Proof. Pass to the completion of R to assume that R is complete. Let $\mathbf{x}$ be a system of parameters for R; this is a maximal R-regular sequence. Using the Theorem of Auslander, Ding and Solberg, there is a one-to-one correspondence between the set of isomorphism classes of semidualizing R-modules and that of $R/\mathbf{x}R$-modules. So we pass to the quotient $R/\mathbf{x}R$ to assume that R is artinian. Since R is equicharacteristic and artinian, Cohen's structure theorem implies R is a k-algebra. Now the theorem follows from a result in representation theory.

General form of Vasconcelos' Conjecture. The number of isomorphism classes of semidualizing modules over a local ring is finite.

To answer this question, we consider Differential Graded algebras and Differential Graded modules (DG algebras and DG modules).

DG Algebras And DG Modules

Complexes of R-modules (R-complexes for short) are indexed homologically:

$$M = \cdots \xrightarrow{\partial^M_{n+2}} M_{n+1} \xrightarrow{\partial^M_{n+1}} M_n \xrightarrow{\partial^M_n} M_{n-1} \xrightarrow{\partial^M_{n-1}} \cdots.$$

and if $m \in M_i$ we write $|m| = i$.

Definition 6. A *commutative differential graded algebra over* R (DG R-algebra for short) is a bounded below ($A_i = 0$ for $i < 0$) R-complex A equipped with a chain map $\mu^A \colon A \otimes_R A \to A$ denoted $\mu^A(a \otimes b) = ab$ (which is called the product) that is

1. **Associative:** for all $a, b, c \in A$ we have $(ab)c = a(bc)$;
2. **Unital:** there is an element $1 \in A_0$ such that for all $a \in A$ we have $1a = a$;
3. **Graded Commutative:** for all $a, b \in A$ we have $ab = (-1)^{|a||b|}ba$ and $a^2 = 0$ when $|a|$ is odd.

The fact that the product on A is a chain map says that ∂^A satisfies the *Leibniz rule*:

$$\partial^A_{|a|+|b|}(ab) = \partial^A_{|a|}(a)b + (-1)^{|a|}a\partial^A_{|b|}(b).$$

Example 7. For elements $a_1, \cdots, a_n \in \mathfrak{m}$, the *Koszul complex* $K = K^R(a_1, \cdots, a_n)$ is the exterior algebra on R^n, considered as an R-complex

$$K = 0 \to K_n \xrightarrow{\partial^K_n} K_{n-1} \xrightarrow{\partial^K_{n-1}} \cdots \xrightarrow{\partial^K_2} K_1 \xrightarrow{\partial^K_1} K_0 \to 0.$$

Here $K_0 = K_n = R$ and $K_1 = R^n$. Given a basis $e_1, \cdots, e_n \in K_1$, for each $2 \leq i \leq n-1$ the module $K_i = \wedge^i(K_1)$ is a free R-module with basis

$$\{e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_i} \mid 1 \leq j_1 \leq j_2 \leq \cdots \leq j_i \leq n\}$$

and the differential ∂_i^K is given as follows:

$$\partial_i^K(e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_i}) = \sum_{z=1}^{i} (-1)^{z+1} a_{j_z}(e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{z-1}} \wedge e_{j_{z+1}} \wedge \cdots \wedge e_{j_i}).$$

Now K is a DG R-algebra with product given by the wedge product.

Remark 8. The natural map $R \to K$ is a "morphism of DG R-algebras". This morphism is our substitute for the natural surjection $R \to R/\mathbf{x}R$.

Facts 9.

1. K is like a flat ring extension of R, and for each i, $\mathrm{H}_i(K)$ is a finite dimensional vector space over the residue field k.
2. $K \simeq T$ where T is a finite dimensional DG algebra over k.

Definition 10. Let A be a DG R-algebra. A *differential graded module over* A (DG A-module for short) is an R-complex M with a chain map $\mu^M \colon A \otimes_A M \to M$ with $am := \mu^M(a \otimes m)$. The map μ^M is the scalar product on M. The fact that the scalar product on M is a chain map says that ∂^M satisfies the Leibniz rule:

$$\partial^M_{|a|+|m|}(am) = \partial^A_{|a|}(a)m + (-1)^{|a|}a\partial^M_{|m|}(m)$$

for all $a \in A$, $m \in M$. A bounded below DG A-module N is *semifree* if its underlying $A^\natural$-module $N^\natural$ has a basis $\{e_\lambda\}_{\lambda \in \Lambda}$.

Facts 11.

1. If M is an R-complex, then $K \otimes_R M$ is a DG K-module.
2. $K \simeq T$ implies that DG K-modules are in one-to-one correspondence with DG T-modules.

Definition 12. Let A be a "noetherian" DG R-algebra, and let M be a DG A-module. Then M is a *semidualizing DG A-module* if M is homologically finite and the homothety morphism $\chi^M \colon A \to \mathbf{R}\mathrm{Hom}_A(M, M)$ induced by $X^M \colon A \to \operatorname{Hom}_A(M, M)$ with $X^M(a)(m) = am$ is a quasiisomorphism. A *semidualizing R-complex* is a semidualizing DG R-module.

Facts 13.

1. M is a semidualizing R-module if and only if $K \otimes_R M$ is a semidualizing DG K-module.
2. If M and M' are semidualizing R-modules and $K \otimes_R M \simeq K \otimes_R M'$, then $M \cong M'$.

Solution To The General Form Of The Conjecture

Let ℓ be an algebraically closed field and

$$A = 0 \longrightarrow \ell^{\alpha_m} \xrightarrow{\partial_m} \ell^{\alpha_{m-1}} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_1} \ell^{\alpha_0} \longrightarrow 0$$

be a finite dimensional commutative DG algebra over ℓ. Also let $W = \bigoplus_{i=0}^{s} W_i$ be a graded module with $\dim_\ell(W_i) = r_i$ for every $0 \leq i \leq s$. Let $\mathbf{r} = (r_0, \cdots, r_s)$ and $\operatorname{Mod}^A_{\mathbf{r}}$ be the set of all DG A-module structures on W. $\operatorname{Mod}^A_{\mathbf{r}}$ is the set of rational points of an algebraic scheme $\underline{\operatorname{Mod}}^A_{\mathbf{r}}$ over ℓ, which is described in the functorial point of view as follows; for any finite dimensional commutative DG algebra A' over ℓ, we have

$$\underline{\operatorname{Mod}}^A_{\mathbf{r}}(A') = \Big\{\text{DG } A' \otimes_\ell A\text{-module structures on } A' \otimes_\ell W\Big\}.$$

Theorem 14. There are only finitely many isomorphism classes of semidualizing DG A-modules with the fixed dimension vector $\mathbf{r} = (r_0, \cdots, r_s)$.

Sketch of Proof. It can be seen that the orbit of every semidualizing DG A-module in $\operatorname{Mod}^A_{\mathbf{r}}$ under $\underline{\mathrm{GL}}(W)$ is an open subscheme of $\underline{\operatorname{Mod}}^A_{\mathbf{r}}$. Let $\{u_\lambda\}_{\lambda \in \Lambda}$ be the set of these open sets, which are disjoint. $\{u_\lambda\}_{\lambda \in \Lambda}$ is an open cover for the open subset $\bigcup_{\lambda \in \Lambda} u_\lambda$ of $\underline{\operatorname{Mod}}^A_{\mathbf{r}}$. Thus it has a finite subcover. Since the open sets u_λ's are disjoint, Λ is a finite set.

Theorem 15. Let B be a DG R-algebra such that each B_i is free over R of finite rank. For an element $t \in \mathfrak{m}$, let $K^R(t)$ be the Koszul complex $0 \to K_1 \xrightarrow{t} K_0 \to 0$ and let D denote the DG R-algebra $K^R(t) \otimes_R B$. If R is complete and N is a semifree DG D-module such that $\operatorname{Ext}_D^2(N, N) = 0$, then N is liftable to B.

Theorem 16. Let R be complete and assume that B and D are the DG R-algebras considered in Theorem 15. Let N be a semifree DG D-module such that $\operatorname{Ext}_D^d(N, N) = 0$ for some even integer d. If M is a lifting of N to B, then $\operatorname{Ext}_B^d(M, M) = 0$.

Theorem 17. Suppose that R is complete and $\underline{t} = t_1, \cdots, t_n$ is a sequence of elements of $\mathfrak{m}$. If N is a semidualizing DG module over $K := K^R(\underline{t})$, then there exists a semidualizing R-complex M which is a lifting of N to R.

Sketch of Proof. We can replace N by a semifree DG resolution over K and assume that N is semifree DG K-module. The existence of an R-complex M which is a lifting of N to R follows by induction on n from the previous two theorems. But note that

$$\begin{aligned} K \;\simeq\; \mathbf{R}\mathrm{Hom}_K(N, N) &= \mathbf{R}\mathrm{Hom}_K(K \otimes_R M, K \otimes_R M) \\ &\simeq \mathbf{R}\mathrm{Hom}_R(M, K \otimes_R M) \\ &\simeq K \otimes_R \mathbf{R}\mathrm{Hom}_R(M, M), \end{aligned}$$

where the last isomorphism is a special case of tensor-evaluation. Therefore, $R \simeq \mathbf{R}\mathrm{Hom}_R(M, M)$.

Notation 18. Let A be a commutative local DG algebra. $\mathfrak{S}(A)$ (resp. $\mathfrak{S}(R)$) denotes the set of shift-isomorphism classes of semidualizing DG A-modules (resp. semidualizing R-complexes).

Corollary 19. There is a one-to-one and onto map from the set $\mathfrak{S}(R)$ to the set $\mathfrak{S}(K)$.

Theorem 20. There are only finitely many semidualizing R-complexes up to shift isomorphism.

Sketch of Proof. Without loss of generality we may assume that R is complete and k is algebraically closed.

Let $\mathbf{a} = a_1, \cdots, a_d$ be a minimal set of generators for $\mathfrak{m}$ and set $K = K^R(\mathbf{a})$. We know that $K \simeq T$ where T is of the form

$$T = 0 \longrightarrow k^{\alpha_m} \xrightarrow{\partial_m} k^{\alpha_{m-1}} \xrightarrow{\partial_{m-1}} \cdots \longrightarrow k^{\alpha_1} \xrightarrow{\partial_1} k \longrightarrow 0$$

where each α_i is a nonnegative integer. In fact T is a finite dimensional DG algebra over k. Now by Corollary 19 and Fact 11(2) we get the following equivalences

$$\mathfrak{S}(R) \overset{\sim}{\longleftrightarrow} \mathfrak{S}(K) \overset{\sim}{\longleftrightarrow} \mathfrak{S}(T).$$

Now suppose that C is a semidualizing R-complex and denote by $\widetilde{C}$ the semidualizing DG T-module corresponding to C. Without loss of generality we can assume that $\inf(C) = 0$. By using a minimal semifree DG resolution, without loss of generality we can suppose that $\widetilde{C}$ is a semifree DG T-module. We also can replace $\widetilde{C}$ by its left truncation $\tau(\widetilde{C})_{\leq n}$ where $n = \operatorname{cmd} R + \operatorname{edim} R$ and suppose that

$$\widetilde{C} = 0 \to \widetilde{C}_n \to \widetilde{C}_{n-1} \to \cdots \to \widetilde{C}_1 \to \widetilde{C}_0 \to 0$$

where for every i, $\widetilde{C}_i = \bigoplus_{j=0}^{i} T_j^{\beta'_{i-j}}$. Now it can be seen that

$$\sum_{i=0}^{n} \dim_k(\widetilde{C}_i) \leq \operatorname{length}(T)\Big(\sum_{i=0}^{n} \mu_R^{i+\operatorname{depth} R}(R)\Big)$$

where the right-hand side is a constant. It is obvious that there are only finitely many $(r_0, \cdots, r_n) \in \mathbb{N}^{n+1}$ with $\sum_{i=0}^{n} r_i \leq \operatorname{length}(T)\Big(\sum_{i=0}^{n} \mu_R^{i+\operatorname{depth} R}(R)\Big)$. Therefore, Theorem 14 implies that $\mathfrak{S}(T)$ and hence $\mathfrak{S}(R)$ is finite.