Math 720 – Preliminary Exam

August 2024

Time: 120 mins.

1. Write your student ID number at the top of each page of your exam solutions.
2. Write only on the front page of each solution sheet.
3. Start each question on a new sheet of paper. Each question is worth 10 points.
4. In answering any part of a question, you may assume the results in the previous parts.
5. To receive full credit, answers must be justified.
6. You can do the problems in any order! If you get stuck, move on and come back to it.
7. In this exam, "ring" means "ring with unit" and "module" means "unital (unitary) module". Further, if $\phi : R \to S$ is a ring homomorphism, we assume $\phi(1_R) = 1_S$.

Student ID Number: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total:	70	

1. (10 points) (a) Consider the function $f_A : \mathbb{Z}^3 \to \mathbb{Z}^3$ given by left multiplication by the matrix
$$A = \begin{bmatrix} 0 & 3 & 6 \\ 1 & -2 & -6 \\ 2 & 5 & 0 \end{bmatrix}.$$
Compute the $\mathbb{Z}$-module structure of $\mathbb{Z}^3/\mathrm{Im}(f_A)$, i.e., express $\mathbb{Z}^3/\mathrm{Im}(f_A)$ as a direct sum of cyclic $\mathbb{Z}$-modules (using its invariant factors).

   (b) Now consider the similar function $h_A : \mathbb{Q}^3 \to \mathbb{Q}^3$ given by left multiplication by the matrix
$$A = \begin{bmatrix} 0 & 3 & 6 \\ 1 & -2 & -6 \\ 2 & 5 & 0 \end{bmatrix}.$$
Compute the Jordan canonical form of A over $\mathbb{Q}$ or show one does not exist. Make sure to fully justify your answer.
Hint: It may help to know that 1 *is an eigenvalue*

2. (10 points) Let k be a field and consider a linear map $g : k^6 \to k^6$ given by a matrix B. Give all possible rational canonical forms of B given the following information.
   - The minimal polynomial of B is $(x-1)^2(x+2)$.
   - Three of the invariant factors of B are constant.

3. (10 points) Let I and J be ideals of a commutative ring R.
   (a) Prove that every element of $M = R/I \otimes_R R/J$ can be written as a simple tensor of the form $[1]_I \otimes [r]_J$ where $[-]_I$ denotes the equivalence class in R/I (and similarly for J).
   (b) Prove that the R-module homomorphism $f : M \to R/(I+J)$ given by $[1]_I \otimes [r]_J \mapsto [r]_{I+J}$ is an isomorphism.

4. (10 points) Let R be a PID and suppose Q is an R-module. Prove that Q is injective if and only if $rQ = Q$ for every nonzero $r \in R$.

5. (10 points) (a) Compute all maximal ideals of $R = k[x]/(x^3 - 1)$ where $k = \mathbb{Z}/2\mathbb{Z}$.
   (b) Using Part (a), give all fields F, up to isomorphism, for which there exists a surjective ring homomorphism $f : R \to F$.

6. (10 points) Prove that $\mathbb{Z}[\sqrt{-7}]$ is not a UFD.

7. (10 points) An ideal I of a commutative ring R is called *primary* if whenever $xy \in I$ then $x \in I$ or $y^n \in I$ for some positive integer n.

   Suppose R is a PID and $I \subseteq R$ is an ideal. Prove that I is primary if and only if $I = P^n = P \cdot P \cdots P$ for some positive integer n and some prime ideal P.