Preliminary Examination (Math 720)
January 2023

Instructions:

- Write your student ID number at the top of each page of your exam solution.
- Write only on the front page of your solution sheets.
- Start each question on a new sheet of paper. Each question is worth 10 points.
- In answering any part of a question, you may assume the results in previous parts.
- To receive full credit, answers must be justified.
- In this exam "ring" means "ring with identity" and "module" means "unital (unitary) module". If $\varphi : R \to S$ is a ring homomorphism, we also assume $\varphi(1_R) = 1_S$

1. Let $R = \mathbb{Z}[i\sqrt{5}]$.

(a) Show that 3 is an irreducible element of R.

(b) Prove that the elements 6 and $2 + 2i\sqrt{5}$ do not have a greatest common divisor.

2. Let R and S be two commutative rings and let K be an ideal of the ring $R \times S$. Prove that there exist ideals I of R and J of S such that $K = I \times J$.

3. Let R be a commutative ring and let M, N be R-submodules of an R-module L. Prove that if $M + N$ and $M \cap N$ are finitely generated, then so are M and N.

4. Let R be a commutative ring and let F be a free R-module of rank n. If $x_1, x_2, \ldots, x_n \in F$ generate F, prove that $x_1, \ldots, x_n$ form a basis of F.

5. Prove that there exists an isomorphism of rings

$$\mathbb{Z}[X]/(X^2 + 4, X^2 + 9) \cong \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$$

6. Let G be $\mathbb{Z}$-module generated by v_1, v_2, v_3 subject to the relations

$$\begin{aligned} 6v_1 + 4v_2 + 2v_3 &= 0 \\ -2v_1 + 2v_2 + 6v_3 &= 0 \end{aligned}$$

Prove that $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}$.

7. Let $T : \mathbb{C}^5 \to \mathbb{C}^5$ be a linear operator with characteristic polynomial

$$c_T(X) = (X - 7)^3(X - 9)^2.$$

Find all the possible Jordan canonical forms of T (up to a permutation of the Jordan blocks).