Instructions. Answer **any 4 short questions, and any 4 long questions.** Clearly mark which questions you wish to be graded on this sheet, or else 1-4 and 6-9 will be graded. Show all work, and explain your answers clearly. Solutions will be graded on correctness and clarity. All answers should include some explanation.

Shorter questions: (5 points each)

1. How many ways can 80 identical black blocks and 50 identical white blocks be arranged in a line so that no two white blocks are adjacent?
2. What is the number of compositions of n into parts that are each even?
3. How many permutations on [4] avoid the pattern 123?
4. How many functions are there from [5] to [8] that hit 2 but do not hit 6?
5. How many pairs of sets (A, B) are there where $A \cup B = [10]$? (If $A = [10]$ and $B = \{4\}$, this is not the same as if $A = \{4\}$ and $B = [10]$.)

Longer questions: (10 points each)

6. Give a combinatorial proof of the inclusion/exclusion formula. That is: let S be a set of size N, and let $\{c_i\}_{i=1}^t$ be a set of properties. Let $N(c_1)$ be the number of elements from S satisfying property c_1, let $N(c_1c_5)$ be the number of elements from S satisfying properties c_1 and c_5, and so on. Show that the number of elements from S satisfying no properties is given by

   $$N-[N(c_1)+\cdots+N(c_t)]+[N(c_1c_2)+N(c_1c_3)+\cdots+N(c_{t-1}c_t)]-\cdots+(-1)^tN(c_1c_2\cdots c_t).$$

   To do so, consider an element x satisfying exactly k properties, and show how many times it is counted on each side of the equation.
7. What is the number of lattice paths from $(0, 1)$ to $(8, 10)$ that never *touch* the line $y = x$?
8. Use generating functions to prove that the number of partitions of n where each part occurs at most k times is equal to the number of partitions of n where each part is not divisible by $k + 1$. In your proof, explain how you derived your generating functions.
9. How many injective functions $f : [10] \to [20]$ satisfy $f(i) \neq i$ for $i = 1, \ldots, n$? You may leave your answer as a single summation.
10. Find the generating function for the following sequence:

    $$\{0, 0 + 1, 0 + 1 + 2, 0 + 1 + 2 + 3, \ldots\}$$