ALGEBRA PRELIMINARY EXAMINATION

AUGUST 2004

NOTES. $\mathbb{Z}$ and $\mathbb{Q}$ are the integers and the rational numbers respectively. All rings are commutative with identity unless specifically indicated otherwise.

(1) Let p be a positive prime integer and G a nonabelian group of order p^3 with center $Z(G)$. Show that $G/Z(G) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$.
(2) Show that there is no simple group of order 72.
(3) Let G be a group and N a normal subgroup of G such that G/N is abelian. For some $y \in G$, let $\phi_y : G \longrightarrow G$ be the automorphism of G defined by $\phi_y(x) = y^{-1}xy$ for all $x \in G$. Show that for all $g \in G$ that $g^{-1}\phi_y(g) \in N$.
(4) A *Boolean ring* is a ring with the property that $x^2 = x$ for all $x \in R$. Show that any Boolean ring is commutative.
(5) Let R be a commutative ring and I an ideal of R. Show that if R/I is a projective $R-$module, then I is a principal ideal generated by an idempotent element (that is, an element x such that $x^2 = x$).
(6) Let R be commutative with identity. An ideal $I \subseteq R$ is said to be idempotent if $I^2 = I$. Show that R contains a proper ideal that is maximal with respect to being idempotent.
(7) Find all possible Jordan canonical forms of a 4×4 real matrix, A, such that $A^3 = 0$
(8) Let J be an injective $\mathbb{Z}-$module and M any $\mathbb{Z}-$module. Show that $J \otimes_{\mathbb{Z}} M$ is an injective $\mathbb{Z}-$module (hint: what is an equivalent characterization of an injective $\mathbb{Z}-$module?).
(9) Consider the following commutative diagram of $R-$modules with both rows exact (you may assume that R is commutative with 1 and that all modules are unitary).

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & 0 \end{array}$$

Show that if f and h are surjective, then so is g.
(10) Let $\xi_5 = e^{(\frac{2\pi i}{5})}$ be a primitive 5^{th} root of unity. Consider the polynomial $f(x) = x^5 + 7$. Compute the Galois group of this polynomial over the fields $\mathbb{Q}(\xi_5)$ and $\mathbb{R}$.