ALGEBRA PRELIMINARY EXAMINATION

JANUARY 2005

Notes. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are the integers, the rational numbers, the real numbers, and the complex numbers respectively. All rings have identity unless specifically indicated otherwise.

(1) Show that there is no simple group of order 132.

(2) Let G be a (finite) nonabelian simple group and p a positive prime. Show that the intersection of all the Sylow $p-$subgroups of G is the identity.

(3) For a group G suppose that $\chi\colon G \longrightarrow \mathbb{C}^*$ is a group homomorphism. Show that the map χ is constant on congugacy classes of G.

(4) Show that if M is a simple R-module then the ring of R-endomorphisms of M is a division algebra containing R.

(5) Show that if R is a commutative integral domain and $I \subsetneq R$ is a proper ideal of R then R/I is a projective R-module if and only if I is (0).

(6) Show that an element is a unit of a commutative ring if and only if it is not contained in a proper ideal.

(7) Consider the following matrix.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 2 & 0 \end{bmatrix}$$

This matrix induces a $\mathbb{Q}[x]-$module action on $\mathbb{Q}^3$ (via $f(x) \circ \mathbf{v} = f(T)\mathbf{v}$ where $f(x) \in \mathbb{Q}[x]$, $\mathbf{v} \in \mathbb{Q}^3$, and T is the matrix above). Explain why $\mathbb{Q}^3$ is an indecomposable (that is, cannot be decomposed as a direct sum of two proper submodules) $\mathbb{Q}[x]-$module under this action.

(8) Show that if $\mathbb{F}$ is a field such that $\mathrm{Char}(\mathbb{F}) \neq 2$, then $\mathbb{F}[\sqrt{\alpha}, \sqrt{\beta}]$ has Galois group isomorphic to the non-cyclic group of order four if and only if α, β and $\alpha\beta$ are all not squares in $\mathbb{F}$ (α and β are elements of $\mathbb{F}$).

(9) Let R be commutative with identity and S a multiplicatively closed subset of R. Suppose that $I \subseteq R$ is an ideal such that $I \bigcap S = \emptyset$ and I is maximal with respect to this property. Prove that I is prime.

(10) Let P be a finitely generated projective $R-$module. Show that $\mathrm{Hom}_R(P, R)$ is also a projective $R-$module.