Algebra Preliminary Examination
August 2019

Instructions:

- Write your student ID number at the top of each page of your exam solution.
- Write only on the front page of your solution sheets.
- Start each question on a new sheet of paper.
- For this exam you have two options:
  (i) Submit solutions to questions from part A and from part B.
  (ii) Submit solutions to questions from part A and from part C.
- In answering any part of a question, you may assume the results of previous parts.
- To receive full credit, answers must be justified.
- In this exam "ring" means "commutative ring with identity" and "module" means "unital module". If $\varphi : R \to S$ is a ring homomorphism, then $\varphi(1_R) = 1_S$.
- This exam has two pages.

A. Rings, Modules, and Linear Algebra (required)

1. Let $R = \mathcal{C}[0,1]$ be the ring of continuous real-valued functions defined on the closed interval $[0,1]$. **Prove or disprove** each statement.
   (a) R is an integral domain.
   (b) R is a Noetherian ring.
2. Let p be a prime element of the ring R and let R_p denote the localization at the prime ideal (p). Prove that if R is a UFD, then R_p is a PID.
3. Prove that $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}, C) = 0$ for all cyclic groups C.
4. Let R be a PID and let $a \in R$ be a *nonzero nonunit* element.
   (a) Prove that $R/(a)$ is not flat as an R-module.
   (b) Let M be a finitely generated R-module. Prove that M is flat if and only if it is free.
5. Let M be an R-module and let $S \subseteq M$ be a linearly independent subset. Prove that S is contained in a maximal linearly independent subset of M.
6. Are the matrices $A, B \in \mathcal{M}_5(\mathbb{F}_5)$ displayed below similar? Here, $\mathbb{F}_5$ denotes the field of integers $(\operatorname{mod} 5)$.

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

B. Groups, Fields, and Galois Theory (option 1)

1. A subgroup H of a finite group G is called a Hall subgroup if $\gcd(|H|, [G : H]) = 1$. Let N be a normal subgroup of G and let H be a Hall subgroup of G.

   (a) Prove that $H \cap N$ is a Hall subgroup of N and HN/N is a Hall subgroup of G/N.

   (b) Prove that if $|N| = [G : H]$, then there exists a group homomorphism $\theta : H \to \mathrm{Aut}(N)$ such that $G \simeq N \rtimes_\theta H$.

2. Classify all groups of p^2q^2 where p, q are distinct primes such that $p \nmid q^2-1$ and $q \nmid p^2-1$. **Hint:** Start by showing that G is the direct product of two Sylow subgroups.

3. Does there exist a polynomial $f \in \mathbb{Q}[x]$ for which $\mathbb{Q}(\sqrt[4]{2})$ is a splitting field (over $\mathbb{Q}$)? Justify your answer.

4. Let ω be a primitive 7^{th} root of unity. How many subfields lie between $\mathbb{Q}$ and $\mathbb{Q}(\omega)$? Justify your answer.

C. Homological Algebra (option 2)

1. Let R be a commutative ring and $0 \to A \to Q \to C \to 0$ be a short exact sequence of R-modules where Q is injective. Assume that A is not injective. Prove that

$$\mathrm{id}_R\, A = \mathrm{id}_R\, C + 1.$$

2. Let R be a commutative ring and $0 \to A \to B \to C \to 0$ an exact sequence of finitely generated R-modules, and let I be an ideal of R contained in its Jacobson radical. Prove that $\mathrm{depth}_I\, C \geq \min\{\mathrm{depth}_I\, A - 1, \mathrm{depth}_I\, B\}$.

3. Let A and B be two abelian groups with $mA = nB = 0$ for some relatively prime integers m and n. Prove that

$$\mathrm{Tor}_1^{\mathbb{Z}}(A, B) = 0.$$

4. Let I be a proper ideal in a commutative local ring R with maximal ideal $\mathfrak{m}$. Prove that there exists an isomorphism of R-modules

$$\mathrm{Tor}_1^R(R/I, R/\mathfrak{m}) \cong \frac{I}{I\mathfrak{m}}.$$