Instructions:
- Write your student ID number at the top of each page of your exam solution.
- Write only on the front page of your solution sheets.
- Start each question on a new sheet of paper. Each question is worth 10 points.
- In answering any part of a question, you may assume the results in previous parts.
- To receive full credit, answers must be justified.
- In this exam “ring” means “ring with identity” and “module” means “unital (unitary) module”. If \( \varphi : R \to S \) is a ring homomorphism, we also assume \( \varphi(1_R) = 1_S \).

1. Let \( R \) be an integral domain. Prove that if every descending chain

\[(r_1) \supseteq (r_2) \supseteq (r_3) \supseteq \ldots\]

of principal ideals in \( R \) stabilizes, then \( R \) is a field.

2. Let \( R \) be an integral domain and let \( \pi \in R \) be an irreducible element that is not prime.

   (a) Prove that there exists an element \( r \in R \) such that the 2-generated ideal \( (\pi, r) \) is not a principal ideal. **Hint:** There exist elements \( a, b \in R \) such that \( ab | \pi \) but \( \pi \nmid a \) and \( \pi \nmid b \).

   (b) Construct a 2-generated ideal in the ring \( \mathbb{Z}[\sqrt{-5}] \) that is not principal.

3. Suppose that \( g, h \in \mathbb{Q}[X] \). Use Gauss’s Lemma to prove that if \( f = gh \) belongs to \( \mathbb{Z}[X] \), then the product of any coefficient of \( g \) with any coefficient of \( h \) belongs to \( \mathbb{Z} \).

4. Let \( A, B \) be finite rings with at least two elements and let \( R = A \times B \). Hence, \( A \) is an \( R \)-module via the natural ring homomorphism \( R \to A \) projecting onto the first coordinate. Prove or disprove each statement:

   (a) \( A \) is a projective \( R \)-module.

   (b) \( A \) is a free \( R \)-module.

5. Let \( R \) be an integral domain with field of fractions \( K \) and let \( I \) be an ideal of \( R \). Prove that \( (R/I) \otimes_R K = 0 \) if and only if \( I \neq 0 \).

6. Let \( F \) be a field and let \( U, V, W \) be finite dimensional \( F \)-vector spaces with linear transformations \( S: U \to V \) and \( T: V \to W \). Prove that

\[
\dim_F(\ker(T \circ S)) \leq \dim_F(\ker T) + \dim_F(\ker S)
\]

7. Let \( \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z} \) be the field with two elements and let \( V \) be an \( \mathbb{F}_2 \)-vector space such that \( \dim(V) = 3 \). Find all possible rational canonical forms for a linear transformation \( T: V \to V \) satisfying \( T^6 = 1 \).