Preliminary Examination (Math 721)  
August 2022

Instructions:

- Write your student ID number at the top of each page of your exam solution.
- Write only on the front page of your solution sheets.
- Start each question on a new sheet of paper.
- In answering any part of a question, you may assume the results of previous parts.
- To receive full credit, answers must be justified.

1. Recall that $D_n$ is the dihedral group acting on $n$ vertices. Prove that $D_n$ has a subgroup of order 4 if and only if $n$ is even.

2. We call a subgroup $N \subseteq G$ a maximal normal subgroup provided (i) $N$ is a proper normal subgroup of $G$, and (ii) if $H$ is another normal subgroup of $G$ such that $N \subseteq H \subseteq G$, then either $N = H$ or $H = G$.

   Prove the following statements:
   
   (a) $N$ is a maximal normal subgroup of $G$ if and only if $G/N$ is a simple group.
   (b) If $M$ and $N$ are distinct maximal normal subgroups of $G$, then $M \cap N$ is a maximal normal subgroup of both $M$ and $N$.

3. Classify all groups of order 45.

4. Let $N$ be a normal subgroup of the group $G$.

   (a) Prove that the map $G \times N \to N$ given by $g \cdot n = gng^{-1}$ is a well-defined group action.
   (b) Suppose now that $G$ is a finite $p$-group and that $N$ is a nontrivial normal subgroup. Prove that $N \cap Z(G)$ is a nontrivial subgroup of $G$. Here, $Z(G)$ denotes the center of $G$.

5. Let $F \subseteq K$ be an extension of fields. Prove or disprove each statement:

   (a) If $[K : F]$ is prime, then $K = F(u)$ for every element $u \in K - F$.
   (b) If $[K : F]$ is prime, $F \subseteq K$ is a Galois extension.

6. Recall that a field $F$ is called perfect if every irreducible polynomial $p(x) \in F[x]$ is separable (i.e., has no repeated roots). Prove that if $F$ is a perfect field and the extension $F \subseteq K$ of fields is algebraic, then $K$ is perfect.

7. Let $f(x) \in \mathbb{Q}[x]$ be a cubic polynomial with splitting field $K$ (over $\mathbb{Q}$). Prove that if $\text{Gal}(K/\mathbb{Q}) \simeq \mathbb{Z}_3$, then all the roots of $f$ must belong to $\mathbb{R}$. 