**Instructions.** Answer any 4 short questions, and any 4 long questions. Clearly mark which questions you wish to be graded on this sheet, or else 1-4 and 6-9 will be graded. Show all work, and explain your answers clearly. Answers will be graded on correctness and clarity. All solutions should include some explanation.

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**Shorter questions:** (5 points each)

1. Let \( n \) denote an antichain poset with \( n \) elements. Write a formula for the number of linear extensions of the ordinal sum poset \( n \oplus m \oplus p \).

2. Is the pentagon poset \( a \preceq b \preceq c \preceq d, a \preceq e \preceq d \) a distributive lattice? Why or why not?

3. Given the following two semistandard Young tableaux, \( T = \begin{array}{ccc} \hline 1 & 3 & 3 \\ 2 & 4 \hline \end{array} \) and \( U = \begin{array}{c} 1 \\ 2 \end{array} \), find their product, \( T \ast U \), in the plactic monoid.

4. Compute the Schur symmetric function \( s_{2,2} \) in three variables \( x_1, x_2, x_3 \).

5. Write the basis element of the Specht module corresponding to the standard Young tableau \( \begin{array}{ccc} 1 & 2 & 3 \\ 4 \hline 5 \end{array} \). You do not need to simplify your answer.

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**Longer questions:** (10 points each)


7. For which positive integers is there no connected poset \( P \) with exactly \( n \) chains? Justify your answer.

8. Let \( L(w, k) \) denote the largest sum of the lengths of \( k \) disjoint weakly increasing subsequences of \( w \). Prove the following statement: If \( w \) and \( w' \) are Knuth equivalent words, then \( L(w, k) = L(w', k) \).

Recall the elementary Knuth transformations are:

(a) \( yzx \sim yxz, x < y \leq z \)

(b) \( xzy \sim zxy, x \leq y < z \)

9. Let \( \Lambda_\mathbb{Q} \) denote the ring of homogeneous symmetric functions over \( \mathbb{Q} \). Let \( \omega : \Lambda_\mathbb{Q} \to \Lambda_\mathbb{Q} \) be defined by how it acts on the Schur function basis: \( \omega(s_\lambda) = s_{\lambda'} \), where \( \lambda' \) is the conjugate of the partition \( \lambda \).

(a) What is \( \omega^2 \)? Why?

(b) Define the hall inner product of symmetric functions by: \( \langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu} \), where \( \delta_{\lambda,\mu} = 1 \) if \( \lambda = \mu \) and 0 otherwise. Show that \( \omega \) is an isometry, meaning for symmetric functions \( f, g \in \Lambda_\mathbb{Q}, \langle f, g \rangle = \langle \omega(f), \omega(g) \rangle \).

10. Find the table of characters for the irreducible representations of \( S_3 \). Give reasons for each character value; writing the table from memory without justifying your computations will receive no credit.