ALGEBRA PRELIMINARY EXAMINATION

SUMMER 2001

NOTES. $\mathbb{Z}$ and $\mathbb{Q}$ refer to the integers and the rational numbers respectively. All rings are commutative with identity unless specifically noted otherwise. The word "domain" means "integral domain," and "PID" means "principal ideal domain."

(1) Let G be a finite group of order $2n$. Show that if H is a subgroup of G of order n, then H is normal in G.

(2) Show that S_5 has no subgroup of order 30.

(3) Determine the smallest *odd* integer n such that there is a nonabelian group of order n.

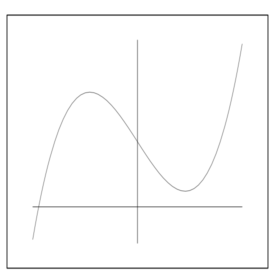
(4) Let R be an integral domain with quotient field K. Assume that R has the property that for all nonzero $\alpha \in K$, either α or α^{-1} is an element of R. Prove the following:
   a) If $a, b \in R$ are nonzero, then either a divides b or b divides a (in R).
   b) Any *finitely generated* ideal of R is principal.

(5) Let R be a commutative ring with identity. Show that any free $R-$module is projective.

(6) Let R be a commutative ring with identity and let $\wp \subseteq R$ be a prime ideal. Show that if $R/\wp$ is a finite ring, then $\wp$ is a maximal ideal.

(7) Let F be an algebraic extension of the field K and let D be an integral domain with $K \subseteq D \subseteq F$. Prove that D is a field.

(8) Find all transitive subgroups of S_3. Use this result to determine the Galois group of an irreducible cubic polynomial in $\mathbb{Q}[x]$ whose graph is pictured below:



(9) Show that the polynomial $f(x) = x^4 + x^3 + x^2 + x + 1$ is irreducible in $\mathbb{Q}[x]$.

(10) Let G be a group of order 2001. Is it possible for G to be simple?