

## ALGEBRA PRELIMINARY EXAMINATION

SPRING 2002

NOTES.  $\mathbb{Z}$  and  $\mathbb{Q}$  refer to the integers and the rational numbers respectively. All rings are commutative with identity unless specifically noted otherwise. The word “domain” means “integral domain,” and “PID” means “principal ideal domain.”

- (1) Let  $p$  and  $q$  be positive primes with  $p > q^2$ . Show that any group of order  $p^2q^2$  is the semidirect product of two abelian groups.
- (2) Show that there is no simple group of order 72.
- (3) Let  $G$  be a finite abelian group. Show that  $G$  is cyclic if and only if  $\text{Aut}(G)$  is abelian.
- (4) Let  $G$  be a group of order  $p^n$  (where  $p$  is a positive prime). Show that the center of  $G$  is nontrivial.
- (5) Let  $R$  be a commutative ring with identity. Let the Jacobson radical,  $J(R)$  be the intersection of all of the maximal ideals of  $R$ . Show that  $x \in J(R) \iff 1 + rx$  is a unit for all  $r \in R$ .
- (6) Show that any injective  $\mathbb{Z}$ -module is a divisible abelian group.
- (7) Show that the multiplicative group of the field of  $p^n$  elements is cyclic.
- (8) Compute the order of the Galois group (over the field of rational numbers) of the polynomial  $x^5 - 3$  and determine whether or not the group is abelian.
- (9) Let  $R$  be a commutative ring with 1 and  $S \subseteq R$  a multiplicatively closed subset of the ring that does not contain 0. Show that there is a prime ideal  $\mathfrak{P} \subseteq R$  such that  $\mathfrak{P} \cap S$  is empty (hint: show that there is an ideal *maximal* with respect to having empty intersection with  $S$ ).
- (10) Show that any maximal ideal of  $R[[x]]$  (with  $R$  commutative with 1) is of the form  $(\mathfrak{M}, x)$  with  $\mathfrak{M}$  a maximal ideal of  $R$ .