Algebra

EXAMINATION

August 2002

Notation. In the following examination p denotes a prime number. The ring of integers is denoted by $\mathbb{Z}$ and the rationals by $\mathbb{Q}$. The letter R will denote a ring, the letter G will denote a group, and A_n will denote the alternating group. Given $f(x) \in R[x]$, we will denote the degree of $f(x)$ by $d(f(x))$.

Problems

(1) Prove that if H and K are finite subgroups of G and $(|H|, |K|) = 1$, then $H \cap K = \{1\}$.

(2) Prove that if K is a characteristic subgroup of H and H is a normal subgroup of G, then K is a normal subgroup of G.

(3) Prove that if H is a normal p-subgroup of a finite group then $H \leq P$ for all P Sylow p-subgroups of G.

(4) Prove that A_n is the only proper normal subgroup of S_n for $n > 4$.

(5) Prove that if $|G| = 300$, then G is not a simple group.

(6) Let F be a field, and $g(x), f(x) \in F[x]$. Prove that if $d(f(x)) \leq d(g(x)) = n$ and if $g(a) = f(a)$ for $n + 1$ distinct elements $a \in F$, then $f(x) = g(x)$.

(7) Show that $g(x) = x^5 - 4x^2 + 2$ is irreducible in $\mathbb{Q}[x]$

(8) Prove that $Z_3[x]/(x^3 - x^2 - 1)$ and $Z_3[x]/(x^3 - x^2 + x - 1)$ are isomorphic.

(9) Let K be a field and R the subring of $K[x]$ consisting of all polynomials having no linear term, prove that R is a domain containing a pair of elements with no gcd.

(10) Prove that the additive group of $\mathbb{Q}$ is not a direct sum of proper subgroups (hint: If $a, b \in \mathbb{Q}$ then there exist $c \in \mathbb{Q}$ such that $a, b \in < c >$).