Algebra

EXAMINATION

January 2003

Notation. In the following examination p denotes a prime number. The ring of integers is denoted by $\mathbb{Z}$ and the rationals by $\mathbb{Q}$. The letter R will denote a ring, the letter G will denote a group, and A_n will denote the alternating group. Given $f(x) \in R[x]$, we will denote the degree of $f(x)$ by $d(f(x))$.

Problems

(1) Prove that if K is a normal subgroup of G and $(|K|, [G : K]) = 1$, then K is the unique subgroup of G having order $|K|$.

(2) Prove that if G is a finite p-subgroup of order p^n, then G has a normal subgroup of order p^k for any $k \leq n$.

(3) Prove that if H is a simple group and $H \leq S_n$ and $|H| > 2$, then H is a subgroup of A_n.

(4) Prove that if $|G| = 96$, then G is not a simple group.

(5) Prove that every Euclidean ring is a PID.

(6) Prove that if $GF(p^n)$ is a subfield of $GF(p^m)$, then $n|m$.(Where $GF(p^n)$ is the Galois field of p^n elements).

(7) Prove that if R is a Euclidean ring and $b \in R$ is neither zero nor a unit, then for any $n \geq 0$ $d(b^n) < d(b^{n+1})$.(Where $d : R \mapsto \mathbb{N} \cup \{0\}$ is the Euclidean function, and you may assume that $d(xy) \geq d(x)$ for nonzero $x, y \in R$).

(8) Let R be a commutative ring with $1 \neq 0$. Suppose that $0 \longrightarrow N \longrightarrow M \longrightarrow (M/N) \longrightarrow 0$ is an exact sequence of R-modules, show that M is a Noetherian R-module if and only if N and M/N are Noetherian R-modules.

(9) Show that the polynomial $x^4 + 1$ is irreducible in $\mathbb{Q}[x]$. (Hint: Consider the transformation $x \longrightarrow x + 1$).

(10) Given that $x^3 + 2x + 1$ is irreducible in $Z_3[x]$, find $(2x + 1)^{-1} \in \frac{Z_3[x]}{(x^3+2x+1)}$.